

Overconfidence and Speculative Bubbles*

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February 2, 2003

Abstract

Motivated by the behavior of asset prices, trading volume and price volatility during historical episodes of asset price bubbles, we present a continuous time equilibrium model where overconfidence generates disagreements among agents regarding asset fundamentals. With short-sale constraints, an asset owner has an option to sell the asset to other overconfident agents when they have more optimistic beliefs. As in Harrison and Kreps (1978), this re-sale option has a recursive structure, that is, a buyer of the asset gets the option to resell it. This causes a significant bubble component in asset prices even when small differences of beliefs are sufficient to generate a trade. In particular, large bubbles are accompanied by large trading volume and high price volatility. Our model has an explicit solution, which allows for several comparative statics exercises. Our analysis shows that while Tobin's tax can substantially reduce speculative trading when transaction costs are small, it has only a limited impact on the size of the bubble or on price volatility. We also give an example where the price of a subsidiary is larger than its parent firm.

*This paper was previously circulated under the title "Overconfidence, Short-Sale Constraints and Bubbles." Scheinkman's research was supported by the Chaire Blaise Pascal and the National Science Foundation. We would like to thank Patrick Bolton, John Cochrane, George Constantinides, Darrel Duffie, Ravi Jagannathan, Owen Lamont, Marcelo Pinheiro, Chris Rogers, Tano Santos, Walter Schachermayer, Harald Uhlig, Dimitri Vayanos, two anonymous referees and seminar participants at various institutions and conferences for comments.

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1 Introduction

The behavior of market prices and trading volumes of assets during historical episodes of price bubbles presents a challenge to asset pricing theories. A common feature of these episodes, including the recent internet stock boom, the tulipmania and the South Sea bubble, is the co-existence of high prices and high trading volume.¹ In addition, high price volatility is frequently observed.²

In this paper, we propose a model of asset trading based on heterogeneous beliefs generated by agents' overconfidence, that generates equilibria that broadly fit these observations. We also provide explicit links between certain parameter values in the model, such as trading cost and information, and the behavior of equilibrium prices and trading volume. In particular, this allows us to discuss the effects of trading taxes and information on prices and volume. More generally, our model provides a flexible framework to study speculative trading that can be used to analyze links between asset prices, trading volume and price volatility.

In the model, the ownership of a share of stock provides an opportunity (option) to profit from other investors' over-valuation. For this option to have value, it is necessary that some restrictions apply to short-selling. In reality, these restrictions arise from many distinct sources. First, in many markets short selling requires borrowing a security and this mechanism is costly.³ In particular the default risk if the asset price goes up is priced by lenders of the security. Second, the risk associated with short selling may deter risk-averse investors. Third, limitations to the availability of capital to potential arbitrageurs may also limit short selling.⁴ For technical reasons, we do not deal with

¹See Garber (2001) for the earlier episodes and Lamont and Thaler (2001), Ofek and Richardson (2001), and Cochrane (2002) for the internet boom. Ofek and Richardson point out on page 1 that "between early 1998 and February 2000, pure internet firms represented as much as 20% of the dollar volume in the public equity market, even though their market capitalization never exceeded 6%."

²Cochrane (2002) refers to the much discussed Palm case on page 6: "Palm stock was tremendously volatile during this period, with 15.4% standard deviation of 5 day returns, which is about the same as the volatility of the S&P 500 index over an entire year"

³Duffie, Garleanu and Pedersen [2002] provide a search model to analyze the actual short-sale process and its implication for asset prices. Jones and Lamont [2002], Geczy, Musto and Reed [2002], and D'Avolio [2002] contain empirical analysis of the relevance of short-sale costs.

⁴Shleifer and Vishny (1997) argue that agency problems limit the capital available to arbitrageurs and may cause arbitrage to fail. See also Xiong (2001), Kyle and Xiong (2001), and Gromb and Vayanos (2002) for studies linking the dynamics of arbitrageurs' capital with asset price dynamics.

short sale costs or risk aversion. Instead we take the extreme view that short sales are not permitted, although our qualitative results would survive the presence of limited short sales as long as the asset owners can expect to make a profit when others have higher valuations.

Our model follows the basic insight of Harrison and Kreps (1978), that, when agents agree to disagree and short selling is not possible, asset prices may exceed their fundamental value. This difference was called the speculative component by Harrison and Kreps. In their model, agents trade because they disagree about the probability distributions of dividend streams. The reason for the disagreement is not made explicit. We study overconfidence, the belief of an agent that his information is more accurate than what it is, as a source of disagreement. Although overconfidence is only one of the many ways by which disagreement among investors may arise - another way is to postulate priors that are not absolutely continuous with respect to each other⁵ - it is suggested by experimental studies of human behavior, and generates a mathematical framework that is relatively easy to treat and allows us to analyze the properties of the equilibrium and to link the dynamics to observables. Our model may also be regarded as a fully worked out example of the Harrison-Kreps framework in continuous time, where computations and comparison of solutions are particularly tractable.

We study a market for a single risky asset with limited supply and many risk-neutral agents in a continuous time model with infinite horizon. The current dividend of the asset is a noisy observation of a fundamental variable that will determine future dividends. In addition to the dividends, there are two other sets of information available at each instant. The information is available to all agents, however, agents are divided in two groups and they differ in the interpretation of the signals. As a consequence, when forecasting future dividends, each group of agents place different weights in the three sets of information, resulting in different forecasts. Although agents in our model know exactly the amount by which their forecast of the fundamental variable exceeds that of agents in the other group, behavioral limitations lead them to agree to disagree. As information flows, the forecasts by agents of the two groups fluctuate, and the group of agents that is at one

⁵As in Morris (1996).

instant relatively more optimistic, may become in a future date less optimistic than the agents in the other group. These changes in relative opinion generate trades.

Each agent in the model understands that the agents in the other group are placing different weights on the different sources of information. When deciding the value of the asset, agents consider their own view of the fundamental as well as the fact that the owner of the asset has an option to sell the asset in the future to the agents in the other group. This option can be exercised at any time by the current owner, and the new owner gets in turn another option to sell the asset in the future. These characteristics makes the option “American” and gives it a recursive structure. The value of the option is the value function of an optimal stopping problem. Since the buyer’s willingness to pay is a function of the value of the option that he acquires, the payoff from stopping is, in turn, related to the value of the option. This gives us a fixed point problem that the option value must satisfy.

We show that when a trade occurs the buyer has the highest valuation of discounted future dividends among all agents, and because of the re-sale option, the price he pays exceeds his valuation of future dividends. Agents pay prices that exceed their own valuation of future dividends, because they believe that in the future they will find a buyer willing to pay even more. This difference between the transaction price and the highest fundamental valuation can be reasonably called a bubble.⁶ A numerical example shows that the magnitude of the bubble component can be large relative to the fundamental value of the asset. Fluctuations in the value of this bubble contribute an extra component to price volatility.

In equilibrium, an asset owner will sell the asset to agents in the other group, whenever his view of the fundamental is surpassed by the view of agents in the other group by a critical amount. We call this difference the critical point. Passages through this critical point determine turnover. When there are no trading costs, we show that the critical point is zero - it is optimal to sell the asset immediately after the valuation of the fundamentals of the asset owner is “crossed” by the valuation of agents in the other

⁶An alternative would be to measure the bubble as the difference between the asset price and the fundamental valuation of the dividends by a rational agent. We opted for our definition because it highlights the difference between beliefs about fundamentals and trading price.

group. Our agents' beliefs satisfy simple stochastic differential equations and it is a consequence of properties of Brownian motion, that once the beliefs of agents cross, they will cross infinitely many times in any finite period of time right afterwards. This results in a trading frenzy, in which the unconditional average volume in any time interval is infinite. Since the equilibria display continuity with respect to the trading cost c , our model is able to capture the excessive trading observed in bubbles with small trading costs.

When trading costs are small, in addition to large volume, the value of the bubble and the extra volatility component are maximized. We show that increases in some parameter values, such as the degree of overconfidence or the information content of the signals, increase these three key variables. In this way, our model provides an explanation for the co-movements of price, volume and volatility observed in actual bubbles.⁷

In the model, increases in trading costs reduce the trading frequency, asset price volatility, and the option value. This effect is very significant for trading frequency when the cost of trading is small. At zero cost, an increase in the cost of trading has an infinite marginal impact in the critical point and in the trading frequency. However, the impact on price volatility and on the size of the bubble is much more modest. As the trading cost increases, the increase in the critical point also raises the profit of the asset owner from each trade, thus partially offsetting the decrease in the value of the re-sale option caused by the reduction in trading frequency. Our analysis suggests that a transaction tax, such as proposed by Tobin (1978), would, in fact, substantially reduce the amount of speculative trading in markets with small transaction costs. However, our analysis also predicts that a transaction tax would have a limited effect on the size of the bubble or on price volatility. Since a Tobin tax will no doubt also deter trading generated by fundamental reasons that are absent from our model,⁸ the limited impact of the tax on the size of the bubble and on price volatility cannot serve as an endorsement of the Tobin tax. The limited effect of transaction costs on the size of the bubble is also compatible

⁷Cochrane (2002) provides direct evidence that prices and volume are correlated in both time series and cross section of US stocks for the period of 1998-2000.

⁸See Dow and Rahi (2000) and references therein for studies of effects of taxes on trading generated by asymmetric information.

with the observation of Shiller (2000) that bubbles have occurred in real estate market, where transaction costs are high.

The existence of the option component in the asset price creates potential violations to the law of one price. Through a simple example, we illustrate that the bubble may cause the price of a subsidiary to be larger than that of its parent firm. The intuition behind the example is that if a firm has two subsidiaries with fundamentals that are perfectly negatively correlated, there will be no differences in opinion, and hence no option component on the value of the parent firm, but possibly strong differences of opinion about the value of a subsidiary. In this example, our model also predicts that trading volume on the subsidiaries would be much larger than on the parent firm. This nonlinearity of the option value may help explain the mispricing of carve-outs that occurred in the late 90's such as the 3Com-Palm case.⁹

Our model often exhibits a stationary bubble and, at first glance, does not seem appropriate to analyze the appearance of bubbles or crashes. In subsection 6.4, we discuss how to accommodate fluctuations in parameter values that can generate fluctuations in the average size of the bubble. This can accommodate crashes and the appearance of bubbles, but does not explain why parameter values fluctuate.

The structure of the paper follows. In Section 2, we present a brief literature review. Section 3 describes the structure of the model. Section 4 derives the evolution of agents' beliefs. In Section 5, we discuss the optimal stopping time problem and derive the equation for equilibrium option values. In Section 6, we solve for the equilibrium. Section 7 discusses several properties of the equilibrium dynamics when trading costs are small. In Section 8, we focus on the effect of trading costs on the equilibrium dynamics. In Section 9, we construct an example where the price of a subsidiary is larger than its parent firm. Section 10 concludes with some discussion of corporate strategies that may be justified in the presence of overconfidence and would not be rewarding in the absence of heterogeneous beliefs.

⁹Lamont and Thaler (2001), Mitchell, Pulvino and Stafford (2001), and Schill and Zhou (2000) empirically analyze mispricings and trading volume in recent carve-outs. In particular, Lamont and Thaler (2001) remarked that the turnover rate of the subsidiaries' stocks was on average six times higher than that of the parent firms' stocks, consistent with our model.

2 Related literature

There is a large literature on the effects of heterogeneous beliefs. In a static framework, Miller (1977), and Chen, Hong and Stein (2002) point out that when investors have heterogeneous beliefs, assets will be held by those with highest beliefs and, if short sales are ruled out and beliefs are unbiased, this will produce overvaluation of assets. This static framework cannot generate an option value. Harris and Raviv (1993) use heterogeneous beliefs in a dynamic model to generate trading. In their model, prices always equal the discounted payoffs expected by a fixed group of agents, and thus there is no option value and no bubble. Kandel and Pearson (1995) study a variation of the Harris-Raviv model and also provide some empirical evidence that heterogeneous beliefs is a driving force for trading. Kyle and Lin (2002) study the trading volume caused by overconfident traders in a model without short-sale constraints and hence no option value or bubbles.

Psychology studies suggest that people may display overconfidence in some circumstances. Alpert and Raiffa (1982), and Brenner et al. (1996) find that subjects overestimate the precision of their knowledge, especially for challenging judgement tasks (Lichtenstein, Fischhoff, and Phillips (1982)). Camerer (1995) argues that even experts can display overconfidence. A similar phenomena is *illusion of knowledge* the fact that persons who do not agree become more polarized when given arguments that serve both sides (Lord, Ross and Lepper (1979)). Hirshleifer (2001) and Barber and Odean (2002) contain reviews of the literature.

In finance, researchers have developed theoretical models to analyze the implications of overconfidence on financial markets. Odean (1998) demonstrates that overconfidence causes excessive trading in a static asymmetric information model. Kyle and Wang (1997) show that overconfidence can be used as a commitment device over competitors to improve one's welfare. Daniel, Hirshleifer and Subrahmanyam (1998) use overconfidence to explain the predictable returns of financial assets. Bernardo and Welch (2001) discuss the benefits of overconfidence to entrepreneurs through the reduced tendency to herd. In these studies, overconfidence is modelled as overestimation of the precision of

one’s information. We follow a similar approach, but emphasize the speculative motive generated through overconfidence in this paper.

The bubble proposed in our model, based on the recursive expectations of traders to take advantage of the mistakes of each other, is very different from the rational bubbles studied in the previous literature including Blanchard and Watson (1982) and Santos and Woodford (1997). In contrast to our set up, these models are incapable of connecting bubbles with large volumes of trade. In addition, since all investors in the models of rational bubbles have the same rational expectations, assets must have infinite maturity to generate bubbles. In our case although we chose, for mathematical simplicity, to treat the infinite horizon case the bubble in our model does not require infinite maturity. If an asset has a finite maturity the bubble will tend to diminish as maturity approaches, but it would nonetheless exist in equilibrium.

Other mechanisms have been proposed to generate asset price bubbles, *e. g.*, Allen and Gorton (1993) through agency problem, Allen, Morris, and Postlewaite (1993) through higher order beliefs, Horst (2001) using social interaction among agents, and Duffie, Garleanu, and Pedersen (2002) using fees from lending stocks to short-sellers. None of these models emphasize the joint dynamics of bubble and trading volume observed in historical episodes.

3 The model

There exists a single risky asset with a dividend process that is the sum of two components. The first component is the fundamental variable that will determine future dividends. The second is “noise”. More precisely, the cumulative dividend process D_t satisfies:

$$dD_t = f_t dt + \sigma_D dZ_t^D, \quad (1)$$

where Z^D is a standard Brownian motion and σ_D is a constant volatility parameter. The fundamental variable f is not observable. However, it satisfies:

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f, \quad (2)$$

where $\lambda \geq 0$ is the mean reversion parameter, \bar{f} is the long-run mean of f , σ_f is a constant volatility parameter and Z^f is a standard Brownian motion. The asset is in finite supply and we normalize the total supply to unity.

There are two sets of risk-neutral agents. The assumption of risk neutrality not only simplifies many calculations, but also serves to highlight the role of information in the model. Since our agents are risk-neutral, the dividend noise in equation (1) has no direct impact in the valuation of the asset. However, the presence of dividend noise makes it impossible to infer f perfectly from observations of the cumulative dividend process. Agents will use the observations of D and any other signals that are correlated with f to infer current f and to value the asset. In addition to the cumulative dividend process, all agents observe a vector of signals s^A and s^B that satisfy:

$$ds_t^A = f_t dt + \sigma_s dZ_t^A \quad (3)$$

$$ds_t^B = f_t dt + \sigma_s dZ_t^B, \quad (4)$$

where Z^A and Z^B are standard Brownian motions. We assume that all four processes Z^D , Z^f , Z^A and Z^B are mutually independent.

Agents in group A (B) think of s^A (s^B) as their own signal although they can also observe s^B (s^A). Heterogeneous beliefs arise because each agent believes that the informativeness of his own signal is larger than its true informativeness. Agents of group A (B) believe that innovations dZ^A (dZ^B) in the signal s^A (s^B) are correlated with the innovations dZ^f in the fundamental process, with ϕ ($0 < \phi < 1$) as the correlation parameter. Specifically, agents in group A believe the process for s^A is

$$ds_t^A = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^A. \quad (5)$$

Although the unconditional volatility of the signal s^A is still σ_s in group A agents' mind, the correlation in the innovations causes them to over-react to signal s^A . Similarly, agents in group B believe the process for s^B is

$$ds_t^B = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^B. \quad (6)$$

Lemma 1 below shows that a larger ϕ , increases the precision that agents attribute

to their own forecast of the current level of fundamentals. For this reason we will refer to ϕ as the overconfidence parameter.¹⁰

Each group is large and there is no short selling of the risky asset. We assume the market to be perfectly competitive in the sense that agents in each group value the asset at their reservation price. To value future cash flows, we may either assume that every agent can borrow and lend at the same rate of interest r , or equivalently that agents discount all future payoffs using rate r , and that each class has infinite total wealth. These assumptions will facilitate the calculation of equilibrium prices.

4 Evolution of beliefs

The model described in the previous section implies a particularly simple structure for the evolution of the difference in beliefs among traders in the two groups. The difference in beliefs is a Markov diffusion with a volatility that is proportional to ϕ . (see Proposition 1 below).

Since all variables are Gaussian, the filtering problem of the agents is standard. With Gaussian initial conditions, the conditional beliefs of agents in group $C \in \{A, B\}$ is Normal with mean \hat{f}^C and variance γ^C . We will characterize the stationary solution. Standard arguments¹¹ allow us to compute the variance of the stationary solution and the evolution of the conditional mean of beliefs. The variance of this stationary solution is the same for both groups of agents and equals

$$\gamma \equiv \frac{\sqrt{(\lambda + \phi\sigma_f/\sigma_s)^2 + (1 - \phi^2)(2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2)} - (\lambda + \phi\sigma_f/\sigma_s)}{\frac{1}{\sigma_D^2} + \frac{2}{\sigma_s^2}}. \quad (7)$$

The following lemma justifies using the term "overconfidence" to describe the effect of a positive ϕ . It states that an increase in ϕ increases the precision that agents attribute to their own forecast.

Lemma 1 *The stationary variance γ decreases with ϕ .*

¹⁰In an earlier draft we assumed that agents overestimate the precision of their signal. We thank Chris Rogers for suggesting that we examine this alternative framework.

¹¹*e.g.* section VI.9 in Rogers and Williams (1987) and Theorem 12.7 in Liptser and Shiriyayev (1977)

Proof: See appendix

In addition, the conditional mean of the beliefs of agents in group A satisfies:

$$d\hat{f}^A = -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt) + \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt). \quad (8)$$

Since f mean-reverts, the conditional beliefs also mean-reverts. The other three terms represent the effects of “surprises.” These surprises can be represented as standard mutually independent Brownian motions for agents in group A :

$$dW_A^A = \frac{1}{\sigma_s}(ds^A - \hat{f}^A dt), \quad (9)$$

$$dW_B^A = \frac{1}{\sigma_s}(ds^B - \hat{f}^A dt), \quad (10)$$

$$dW_D^A = \frac{1}{\sigma_D}(dD - \hat{f}^A dt). \quad (11)$$

Note that these processes are only Wiener processes in the mind of group A agents. Due to overconfidence ($\phi > 0$), agents in group A over-reacts to surprises in s^A .

Similarly, the conditional mean of the beliefs of agents in group B satisfies:

$$d\hat{f}^B = -\lambda(\hat{f}^B - \bar{f})dt + \frac{\gamma}{\sigma_s^2}(ds^A - \hat{f}^B dt) + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^B - \hat{f}^B dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^B dt). \quad (12)$$

These surprise terms can be represented as standard mutually independent Wiener processes for agents in group B :

$$dW_A^B = \frac{1}{\sigma_s}(ds^A - \hat{f}^B dt), \quad (13)$$

$$dW_B^B = \frac{1}{\sigma_s}(ds^B - \hat{f}^B dt), \quad (14)$$

$$dW_D^B = \frac{1}{\sigma_D}(dD - \hat{f}^B dt). \quad (15)$$

Again, we emphasize that these processes form a standard 3-d Wiener process only for agents in group B .

Since the beliefs of all agents have constant variance, we will refer to the conditional mean of the beliefs as their *beliefs*. We let g_A and g_B denote the differences in beliefs:

$$g^A = \hat{f}^B - \hat{f}^A \quad (16)$$

$$g^B = \hat{f}^A - \hat{f}^B. \quad (17)$$

The next proposition describes the evolution of these differences in beliefs:

Proposition 1

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A, \quad (18)$$

where

$$\rho = \sqrt{\left(\lambda + \phi \frac{\sigma_f}{\sigma_s}\right)^2 + (1 - \phi^2)\sigma_f^2 \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)}, \quad (19)$$

$$\sigma_g = \sqrt{2}\phi\sigma_f, \quad (20)$$

and W_g^A is a standard Wiener process for agents in group A, and it is independent to innovations to \hat{f}^A .

Proof: see appendix.

Proposition 1 implies that the difference in beliefs g^A follows a simple mean reverting diffusion process in the mind of group A agents. In particular, the volatility of the difference in beliefs is zero in the absence of overconfidence. A larger ϕ leads to greater volatility. In addition, $\frac{-\rho}{2\sigma_g^2}$ measures the pull towards the origin.¹² A simple calculation shows that this *mean-reversion* decreases with ϕ . A positive ϕ causes an increase in fluctuations of opinions and a slower mean-reversion.

In an analogous fashion, for agents in group B, g^B satisfies:

$$dg^B = -\rho g^B dt + \sigma_g dW_g^B, \quad (21)$$

where W_g^B is a standard Wiener process, and it is independent to innovations to \hat{f}^B .

5 Trading

Fluctuations in the difference of beliefs across agents will induce trading. It is natural to expect that investors that are more optimistic about the prospects of future dividends will bid up the price of the asset and eventually hold the total (finite) supply. We will allow for costs of trading - a seller pays $c \geq 0$ per unit of the asset sold. This cost may represent an actual cost of transaction or a tax.

¹²See Conley *et al.* (1997) for an argument that this is the correct measure of mean-reversion.

At each t , agents in group $C = \{A, B\}$ are willing to pay p_t^C for a unit of the asset. The presence of the short-sale constraint, a finite supply of the asset, and an infinite number of prospective buyers, guarantee that any successful bidder will pay his reservation price.¹³ The amount that an agent is willing to pay reflects the agent's fundamental valuation and the fact that he may be able to sell his holdings at a later date at the demand price of agents in the other group for a profit. If we let $o \in \{A, B\}$ denote the group of the current owner, \bar{o} be the other group, and E_t^o be the expectation of members of group o , conditional on the information they have at t , then:

$$p_t^o = \sup_{\tau \geq 0} E_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} dD_s + e^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right], \quad (22)$$

where τ is a stopping time, and $p_{t+\tau}^{\bar{o}}$ is the reservation value of the buyer at the time of transaction $t + \tau$. Note that $p_{t+\tau}^{\bar{o}} - p_{t+\tau}^o - c$ represents the trading profit to the seller.

Since, $dD = \hat{f}_t^o dt + \sigma_D dW_D^o$, we have, using the equations for the evolution of the conditional mean of beliefs (equations (8) and (12) above) that:

$$\int_t^{t+\tau} e^{-r(s-t)} dD_s = \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + M_{t+\tau}, \quad (23)$$

where $E_t^o M_{t+\tau} = 0$. Hence, we may rewrite equation (22) as:

$$p_t^o = \max_{\tau \geq 0} E_t^o \left\{ \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)} (\hat{f}_t^o - \bar{f})] ds + e^{-r\tau} (p_{t+\tau}^{\bar{o}} - c) \right\}. \quad (24)$$

To characterize equilibria, we will start by postulating a particular form for the equilibrium price function, equation (25) below. Proceeding in a heuristic fashion, we derive properties that our candidate equilibrium price function should satisfy. We then construct a function that satisfies these properties, and verify that in fact we have produced an equilibrium.¹⁴

Since all the relevant stochastic processes are Markovian and time-homogeneous, and traders are risk-neutral, it is natural to look for an equilibrium in which the demand

¹³This observation simplifies our calculations, but is not crucial for what follows. We could partially relax the short sale constraints or the division of gains from trade, provided it is still true that the asset owner expects to make speculative profits from other investors.

¹⁴The argument that follows will also imply that our equilibrium is the only one within a certain class. However, there are other equilibria. In fact, given any equilibrium price p_t^o and a process M_t that is a martingale for both groups of agents, then $\tilde{p}_t^o = p_t^o + e^{rt} M_t$ is also an equilibrium.

price of the current owner satisfies

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + q(g_t^o). \quad (25)$$

with $q > 0$ and $q' > 0$. This equation states that prices are the sum of two components. The first part, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda}$, is the expected present value of future dividends from the view-point of the current owner. The second is the value of the resale option, $q(g_t^o)$, which depends on the current difference between the beliefs of the other group's agents and the beliefs of the current owner. We call the first quantity the owner's fundamental valuation and the second the value of the resale option. Applying equation (25) to evaluate $p_{t+\tau}^o$, and collecting terms, we may rewrite the stopping time problem faced by the current owner, equation (24) as:

$$p_t^o = p^o(\hat{f}_t^o, g_t^o) = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + \sup_{\tau \geq 0} \mathbb{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^o) - c \right) e^{-r\tau} \right]. \quad (26)$$

Equivalently, the resale option value satisfies

$$q(g_t^o) = \sup_{\tau \geq 0} \mathbb{E}_t^o \left[\left(\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^o) - c \right) e^{-r\tau} \right]. \quad (27)$$

Hence to show that an equilibrium of the form (25) exists, it is necessary and sufficient to construct an option value function q that satisfies equation (27). This equation is similar to a Bellman equation. A candidate function q when plugged into the right hand side must yield the same function on the left hand side. The current asset owner chooses an optimal stopping time to exercise his re-sale option. Upon the exercise of the option, the owner gets the "strike price" $\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^o)$, the amount of excess optimism that the buyer has about the asset's fundamental value and the value of the resale option to the buyer, minus the cost c of exercising the option. In contrast to the optimal exercise problem of American options, the "strike price" in our problem depends on the re-sale option value function itself.

It is apparent from the analysis in this section that one could, in principle, treat an asset with a finite life. Equations (22) to (24) would apply with the obvious changes to account for the finite horizon. However, the option value q will now depend on the remaining life of the asset, introducing another dimension to the optimal exercise problem.

The infinite horizon makes the stopping time problem stationary, greatly reducing the mathematical difficulty.

6 Equilibrium

In this section, we derive the equilibrium option value, duration between trades, and contribution of the option value to price volatility. In addition, we also provide a simple way to accommodate crashes.

6.1 Resale option value

Intuitively, the value of the option $q(x)$ should be at least as large as the gains realized from an immediate sale. The region where the value of the option equals that of an immediate sale is the stopping region. The complement is the continuation region. In the mind of the risk neutral asset holder, the discounted value of the option $e^{-rt}q(g_t^o)$ should be a martingale in the continuation region, and a supermartingale in the stopping region.¹⁵ These conditions can be stated as:

$$q(x) \geq \frac{x}{r + \lambda} + q(-x) - c \quad (28)$$

$$\frac{1}{2}\sigma_g^2 q'' - \rho x q' - r q \leq 0, \text{ with equality if (28) holds strictly.} \quad (29)$$

In addition, the function q should be continuously differentiable (smooth pasting). We will derive a smooth function q that satisfies equations (28) and (29) and then use these properties and a growth condition on q to show that in fact the function q solves (27).

To construct the function q , we guess that the continuation region will be an interval $(-\infty, k^*)$, with $k^* > 0$. k^* is the minimum amount of difference in opinions that generates a trade. As usual, we begin by examining the second order ordinary differential equation

¹⁵In equilibrium, risk-neutrality requires that the price of the asset today is not less than the discounted expected price in the future plus the expected discounted dividends that will accrue. In addition, if an agent holds the asset, these two quantities must coincide. According to equation (25), the price of the asset is the sum of two components. The first term, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda}$, is a martingale once we discount and add the expected dividends. Hence the discounted option value must be a martingale in the continuation region and a supermartingale in the stopping region.

that q must satisfy, albeit only in the continuation region:

$$\frac{1}{2}\sigma_g^2 u'' - \rho x u' - r u = 0 \quad (30)$$

The following proposition helps us construct an “explicit” solution to equation (30).

Proposition 2 *Let*

$$h(x) = \begin{cases} U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) & \text{if } x \leq 0 \\ \frac{2\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)} M\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) - U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2\right) & \text{if } x > 0 \end{cases} \quad (31)$$

where $\Gamma(\cdot)$ is the Gamma function, and $M : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $U : \mathbb{R}^3 \rightarrow \mathbb{R}$ are two Kummer functions described in the appendix. $h(x)$ is positive and increasing in $(-\infty, 0)$. In addition h solves equation (30) with

$$h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r}{2\rho}\right)\Gamma\left(\frac{1}{2}\right)}. \quad (32)$$

Any solution $u(x)$ to equation (30) that is strictly positive and increasing in $(-\infty, 0)$ must satisfy: $u(x) = \beta_1 h(x)$ with $\beta_1 > 0$.

Proof: see appendix.

We will also need properties of the function h that are summarized in the following Lemma.

Lemma 2 *For each $x \in \mathbb{R}$, $h(x) > 0$, $h'(x) > 0$, $h''(x) > 0$, $h'''(x) > 0$, $\lim_{x \rightarrow -\infty} h(x) = 0$, and $\lim_{x \rightarrow -\infty} h'(x) = 0$.*

Proof: See appendix.

Since q must be positive and increasing in $(-\infty, k^*)$, we know from Proposition 2 and Lemma 2 that

$$q(x) = \begin{cases} \beta_1 h(x), & \text{for } x < k^* \\ \frac{x}{r+\lambda} + \beta_1 h(-x) - c, & \text{for } x \geq k^*. \end{cases} \quad (33)$$

Since q is continuous and continuously differentiable at k^* ,

$$\beta_1 h(k^*) - \frac{k^*}{r+\lambda} - \beta_1 h(-k^*) + c = 0 \quad (34)$$

$$\beta_1 h'(k^*) + \beta_1 h'(-k^*) - \frac{1}{r+\lambda} = 0. \quad (35)$$

These equations imply that

$$\beta_1 = \frac{1}{(h'(k^*) + h'(-k^*))(r + \lambda)}, \quad (36)$$

and k^* satisfies

$$[k^* - c(r + \lambda)](h'(k^*) + h'(-k^*)) - h(k^*) + h(-k^*) = 0. \quad (37)$$

The next theorem shows that for each c , there exists a unique pair (k^*, β_1) that solves equations (36) and (37). The smooth pasting conditions are sufficient to determine the function q and the “trading point” k^* .

Theorem 1 *For each trading cost $c \geq 0$, there exists a unique k^* that solves (37). If $c = 0$ then $k^* = 0$. If $c > 0$, $k^* > c(r + \lambda)$.*

Proof: see appendix.

When a trade occurs, the buyer has the highest fundamental valuation. The difference between what a buyer pays and his fundamental valuation can be legitimately named a bubble. In our model, this difference is given by

$$b = q(-k^*) = \frac{1}{(r + \lambda)} \frac{h(-k^*)}{(h'(k^*) + h'(-k^*))}. \quad (38)$$

Using equation (38), we can write the value of the re-sale option as

$$q(x) = \begin{cases} \frac{b}{h(-k^*)} h(x), & \text{for } x < k^* \\ \frac{x}{r + \lambda} + \frac{b}{h(-k^*)} h(-x) - c, & \text{for } x \geq k^*. \end{cases} \quad (39)$$

The next theorem establishes that in fact q solves (27). The proof consists of two parts. First, we show that (28) and (29) hold and that q' is bounded. We then use a standard argument¹⁶ to show that in fact q must solve equation (27).

Theorem 2 *The function q constructed above is an equilibrium option value function. The optimal policy consists of exercising immediately if $g^o > k^*$, otherwise wait until the first time in which $g^o \geq k^*$.*

Proof: see appendix.

¹⁶See *e.g.* Kabilia (1993) or Scheinkman and Zariphopoulou (2001) for similar arguments.

6.2 Duration between trades

We let

$$w(x, k, r) = E^o[e^{-r\tau(x,k)}|x], \quad \text{with } \tau(x, k) = \inf\{s : g_{t+s}^o > k\}, \quad \text{given } g_t^o = x \leq k. \quad (40)$$

$w(x, k, r)$ is the discount factor applied to cashflows received the first time that the difference in beliefs reaches the level of k given that the current difference in beliefs is x . Standard arguments¹⁷ show that w is a non-negative and strictly monotone solution to:

$$\frac{1}{2}\sigma_g^2 w_{xx} - \rho x w_x = r w, \quad w(k, k, r) = 1. \quad (41)$$

Therefore, Proposition 2 implies that

$$w(x, k, r) = \frac{h(x)}{h(k)}. \quad (42)$$

Note that the free parameter β_1 does not affect w .

Using the discount factor $w(x, k, r)$, we can interpret the optimal stopping problem in equation (27) as choosing the optimal trading point k^* that solves

$$\sup_{k \geq 0} \left[\left(\frac{k}{r + \lambda} + q(-k) - c \right) w(x, k, r) \right], \quad (43)$$

where x is the current difference in agents' beliefs. The optimal trading point k^* balances the trade-off between larger trading profits $\frac{k}{r+\lambda} + q(-k) - c$ and a smaller discount factor $w(x, k, r)$. Solving this optimization problem gives exactly the same optimal trading point k^* as the one obtained above.

If $c > 0$, trading occurs the first time $t > s$ when $g_t^o = k^*$ given that $g_s^o = -k^*$. The expected duration between trades provides a useful measure of trading frequency. Since w is the moment generating function of τ ,

$$E[\tau(-k^*, k^*)] = - \left. \frac{\partial w(-k^*, k^*, r)}{\partial r} \right|_{r=0}. \quad (44)$$

When $c = 0$, the expected duration between trades is zero. This is a consequence of Brownian local time, as we discuss below.

¹⁷*e.g.* Karlin and Taylor (1981), page 243

6.3 An extra volatility component

The option component introduces an extra source of price volatility. Proposition 1 states that the innovations in the asset owner's beliefs \hat{f}^o and the innovations in the difference of beliefs g^o are orthogonal. Therefore, the total price volatility is the sum of the volatility of the fundamental value in the asset owner's mind, $\frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r+\lambda}$, and the volatility of the option component.

Proposition 3 *The volatility from the option value component is*

$$\eta(x) = \frac{\sqrt{2}\phi\sigma_f}{(r+\lambda)} \frac{h'(x)}{(h'(k^*) + h'(-k^*))}, \quad \forall x \leq k^*. \quad (45)$$

Proof: see appendix.

Since $h' > 0$, and in equilibrium $g^o \leq k^*$, the volatility of the option value is maximum at the trading point $g^o = k^*$.

The volatility of the fundamental value in the asset owner's mind can be derived as

$$\begin{aligned} & \frac{1}{r+\lambda} \left\{ \left(\frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s} \right)^2 + \left(\frac{\gamma}{\sigma_s} \right)^2 + \left(\frac{\gamma}{\sigma_D} \right)^2 \right\} \\ = & \frac{1}{r+\lambda} \frac{1}{(2/\sigma_s^2 + 1/\sigma_D^2)} \left[2\lambda^2 + 2\lambda\phi\sigma_f/\sigma_s + 2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2 \right. \\ & \left. - 2\lambda\sqrt{\lambda^2 + 2\lambda\phi\sigma_f/\sigma_s + (2 - \phi^2)\sigma_f^2/\sigma_s^2 + (1 - \phi^2)\sigma_f^2/\sigma_D^2} \right], \end{aligned}$$

which increases with ϕ if $\lambda > 0$, and becomes $\frac{\sigma_f}{r+\lambda}$ if $\lambda = 0$. Therefore, overconfidence also makes the asset owner's fundamental valuation more volatile. In the remaining part of the paper, we ignore this effect to focus on the extra volatility component caused by the option value.

6.4 Crashes

There are several ways in which we can imagine a change in equilibrium that brings the bubble b to zero. The over-confident agents may correct their over-confidence. The fundamental volatility of the asset may disappear. The public information (the type of

information that all agents can agree on) may become infinitely precise. For concreteness, imagine that agents in group A (B) believe that agents in the other group will at some point change their opinion and agree with them on the precision of the signals s^A and s^B . Suppose further that agents in A (B) believe that this change of mind happens according to a Poisson process Θ^A (Θ^B). Finally, suppose that these Poisson processes have a common Poisson parameter θ and that they are independent of each other and of the four Brownian motions that describe the model. It is easy to see that the option value

$$q(x) = \max_k \left[\left(\frac{k}{r + \lambda} + q(-k) \right) \mathbb{E}_t^o e^{-(r+\theta)\tau} \right]. \quad (46)$$

Effectively, a higher discount rate $r + \theta$ is used for the profits from exercising the option.

More generally, we may postulate that some parameter, σ_f or ϕ , changes according to Poisson times that are independent of all the other relevant uncertainty. The model will then produce results that are qualitatively similar to the case in which these parameters are constant, except that the average size of the bubble at any time will depend on the current value of the parameter. In this way, we can admit the appearance of bubbles and market crashes, although a more interesting discussion should account for reasons for the parameter fluctuations.

In the following sections, we discuss several properties of the equilibrium pricing function and the associated bubble.

7 Properties of equilibria for small trading costs

In this section, we discuss several of the characteristics of the equilibrium dynamics for small trading costs, including the volume of trade and the magnitudes of the bubble and of the extra volatility component caused by the bubble. We also provide some comparative statics and show how parameter changes co-move price, volatility, and turnover.

7.1 Trading volume

It is a property of Brownian motion that if it hits the origin at t , it will hit the origin at an infinite number of times in any non-empty interval $[t, t + \Delta t)$. In our limit case of $c = 0$, this implies an infinite amount of trade in any non-empty interval that contains a

single trade. When the cost of trade $c = 0$, in any time interval, turnover is either zero or infinity, and the unconditional average volume in any time interval is infinity.¹⁸ The expected time between trades depends continuously on c , so it is possible to calibrate the model to obtain any average daily volume. However, a serious calibration would require accounting for other sources of trading, such as shocks to liquidity, and should match several moments of volume, volatility and prices.

7.2 Magnitude of the bubble

When $c = 0$, a trade occurs each time traders' fundamental beliefs "cross". Nonetheless, the bubble is strictly positive, since

$$b = \frac{1}{2(r + \lambda)} \frac{h(0)}{h'(0)}. \quad (47)$$

Owners do not expect to sell the asset at a price above their own valuation, but the option has a positive value. This result may seem counterintuitive. To clarify it, it is worthwhile to examine the value of the option when trades occur whenever the absolute value of the differences in fundamental valuations equal an $\epsilon > 0$. An asset owner in group A (B) expects to sell the asset when agents in group B (A) have a fundamental valuation that exceeds the fundamental valuation of agents in group A (B) by ϵ , that is $g^A = \epsilon$ ($g^A = -\epsilon$). If we write b_0 for the value of the option for an agent in group A that buys the asset when $g^A = -\epsilon$, and b^1 for the value of the option for an agent of group B that buys the asset when $g^A = \epsilon$, then

$$b_0 = \left[\frac{\epsilon}{r + \lambda} + b_1 \right] \frac{h(-\epsilon)}{h(\epsilon)}, \quad (48)$$

where $\frac{h(-\epsilon)}{h(\epsilon)}$ is the discount factor from equation (42). Symmetry requires that $b_0 = b_1$ and hence

$$b_0 = \frac{\epsilon}{(r + \lambda)} \frac{h(-\epsilon)}{[h(\epsilon) - h(-\epsilon)]}. \quad (49)$$

¹⁸The unconditional probability, that it is zero, depends on the volatility and mean reversion of the process of the difference of opinions and on the length of the interval. As the length of the interval goes to infinity, the probability of no trade goes to zero.

Another way of deriving b_0 is to note that by symmetry:

$$b_1 = \left[\frac{\epsilon}{r + \lambda} + b_0 \right] \frac{h(-\epsilon)}{h(\epsilon)}, \quad (50)$$

and hence we may derive an expression for b_0 that reflects the value of all future options to sell, properly discounted:

$$\begin{aligned} b_0 &= \frac{\epsilon}{r + \lambda} \left[\frac{h(-\epsilon)}{h(\epsilon)} + \left(\frac{h(-\epsilon)}{h(\epsilon)} \right)^2 + \left(\frac{h(-\epsilon)}{h(\epsilon)} \right)^3 + \dots \right] \\ &= \frac{\epsilon}{(r + \lambda)} \frac{h(-\epsilon)}{[h(\epsilon) - h(-\epsilon)]}. \end{aligned} \quad (51)$$

As $\epsilon \rightarrow 0$,

$$b_0 \rightarrow \frac{1}{2(r + \lambda)} \frac{h(0)}{h'(0)} = b. \quad (52)$$

In this illustration, as $\epsilon \rightarrow 0$, trading occurs with higher frequency and the waiting time goes to zero. In the limit, traders will trade infinitely often and the small gains in each trade compound to a significant bubble. This situation is similar to the cost from hedging an option using a stop-loss strategy studied in Carr and Jarrow (1990).

It is intuitive that when σ_g becomes larger, there is more difference of beliefs, resulting in a larger bubble. Also, when ρ becomes larger, for a given level of difference in beliefs, the re-sale option is expected to be exercised quicker, and therefore there is also a larger bubble. In fact we can show that:

Lemma 3 *If c is small, the bubble b increases with σ_g and ρ , and decreases with r and θ . For all $x < 0$, $q(x) = b \frac{h(x)}{h(-k^*)}$ increases with σ_g and ρ , and decreases with r and θ .*

Proof: See appendix.

The proof of Lemma 3 actually shows that whenever c is small, the effect of a change in a parameter on the barrier is second order.

Proposition 1 allows us to write σ_g and ρ using the parameters ϕ , λ , σ_f , $i_s = \frac{\sigma_f}{\sigma_s}$, and $i_D = \frac{\sigma_f}{\sigma_D}$. i_s and i_D measure the information in each of the two signals and the dividend flow respectively. To simplify mathematics, we set $\lambda = 0$, then,

$$\sigma_g = \sqrt{2}\phi\sigma_f \quad (53)$$

$$\rho = \sqrt{(2 - \phi^2)i_s^2 + (1 - \phi^2)i_D^2} \quad (54)$$

Differentiating these equations, one can show the following:

As σ_f increases, σ_g increases and ρ is unchanged. Therefore, b and $q(x)$, for $x < 0$, increase. The option value and the bubble increase with the volatility of the fundamental process.

As i_s or i_D increases, σ_g is unchanged and ρ increases, since $0 < \phi < 1$. Therefore, b and $q(x)$, for $x < 0$, increase. The option value and the bubble increase with the amount of information in the signals and the dividend flow.

As ϕ increases, σ_g increases and ρ decreases. Thus, an increase in ϕ has offsetting effects on the size of the bubble. However, numerical exercises indicate that the size of bubble always increases with ϕ .

7.3 Magnitude of the extra volatility component

The volatility of the option value at the trading point is $\frac{\sqrt{2}\phi\sigma_f}{(r+\lambda)} \frac{h'(k^*)}{h'(k^*)+h'(-k^*)}$. Following the proof of Lemma 3, one can establish:

Lemma 4 *If c is small, the volatility of the option value at the trading point decreases with the interest rate r and the degree of mean reversion λ , and increases with the over-confidence parameter ϕ and the fundamental volatility σ_f .*

This Lemma states, in particular, that an increase in the volatility of fundamentals has an additional effect on price volatility at trading points, through an increase in the volatility of the option component.

7.4 Price, volatility and turnover

Our model provides a link between asset prices, price volatility and share turnover. Since these are endogenous variables, their relationship will typically depend on which exogenous variable is shifted. In this section, we illustrate this link using numerical examples with a small trading cost.

Figure 1 shows the effect of changes in ϕ on the equilibrium when there is a small transaction cost on the trading barrier k^* , expected duration between trades, the bubble b , and $\eta(0)$ (the extra volatility component when beliefs coincide). The expected duration

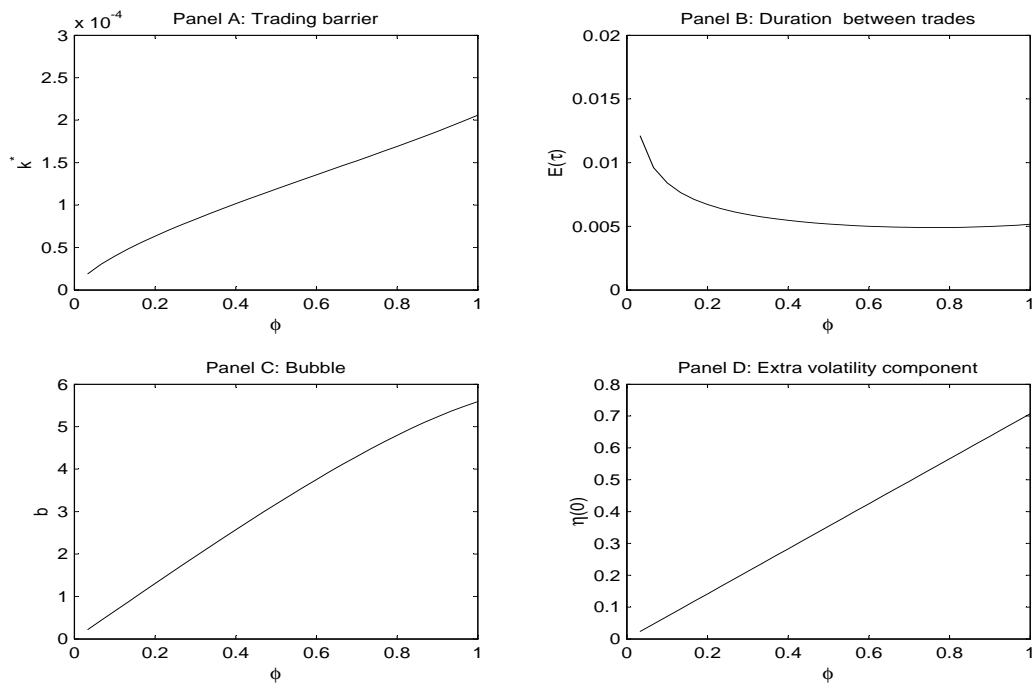


Figure 1: Effects of overconfidence level. The following parameters have been specified: $r = 5\%$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$, $c = 10^{-6}$. The trading barrier, the bubble and the extra volatility component are all measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

between trades is measured in years. The trading barrier, extra volatility $\eta(0)$ and the bubble b are measured in multiples of the fundamental volatility $\frac{\sigma_f}{r+\lambda}$.¹⁹ Recall that, as ϕ increases, the volatility parameter σ_g in the difference of beliefs increases, while the mean reversion parameter ρ decreases. As a result, the resale option becomes more valuable to the asset owner, the bubble and the extra volatility component become larger and the optimal trading barrier becomes higher. The duration between trades is determined by two offsetting effects as ϕ increases. On the one hand, the trading barrier becomes higher making the duration between trades longer. On the other hand, the volatility σ_g of the difference in beliefs increases, causing the duration to be shorter. As we stated, the proof of lemma 3 shows that, when c is small, the change in the trading barrier k^* is second-order. Thus the duration between trades typically decreases, as illustrated in panel B.

Figure 2 shows the effect of changes in the information in signals $i_s = \frac{\sigma_f}{\sigma_s}$ on the equilibrium, again with a small transaction cost. As i_s increases, the mean reversion parameter ρ of the difference in beliefs increases, and the volatility parameter σ_g is unchanged. Intuitively, the increase in ρ causes the trading barrier and the duration between trades to drop. Nevertheless, the bubble becomes larger due to the increase in trading frequency. The extra volatility component η is almost independent of i_s , since it is essentially determined by ϕ and σ_f as shown in equation 45.

In both cases, there is a monotone increasing relationship between the size of bubble and duration between trades. In addition, the extra price volatility does not decrease when one of these other two variables increases. As we mentioned in the introduction and especially as summarized in Cochrane (2002), the positive relationship between bubbles and turnover has been documented in many historical episodes, and there is also a case for high volatility during bubbles. We have also verified that the qualitative relationship that we illustrate here holds for many other parameter values.

¹⁹Since the bubble is generated through an option value, it is natural to normalize it by the volatility of the underlying fundamental value, that is, the price volatility that would prevail if fundamentals were observable.

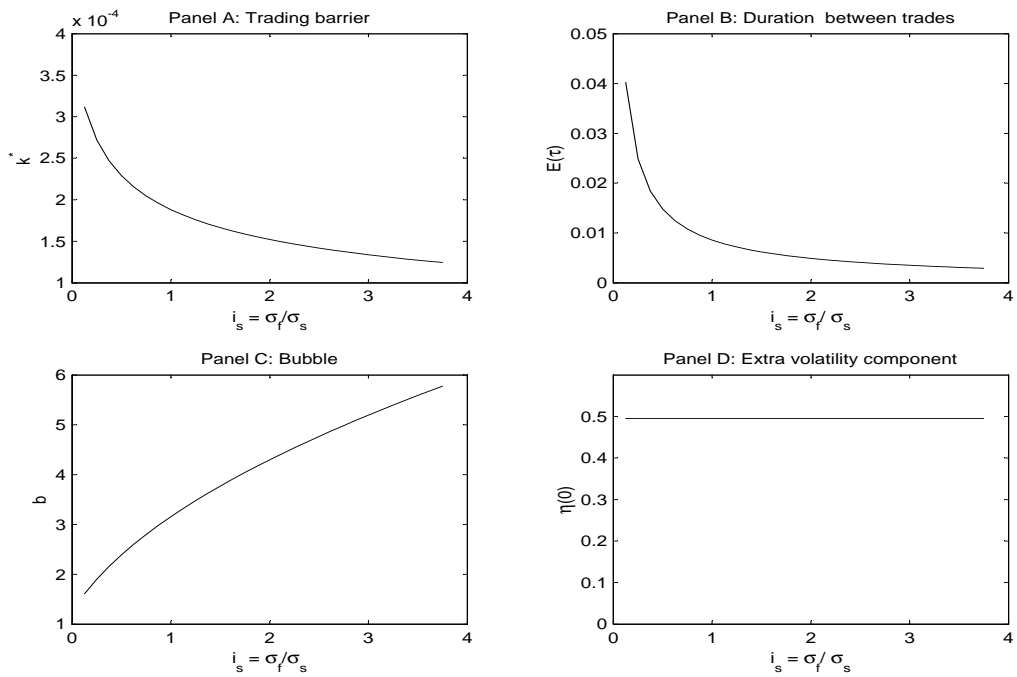


Figure 2: Effects of information in signals. The following parameters have been specified: $r = 5\%$, $\lambda = 0$, $\theta = 0.1$, $\phi = 0.7$, $i_D = 0$, $c = 10^{-6}$. The trading barrier, the bubble and the extra volatility component are all measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

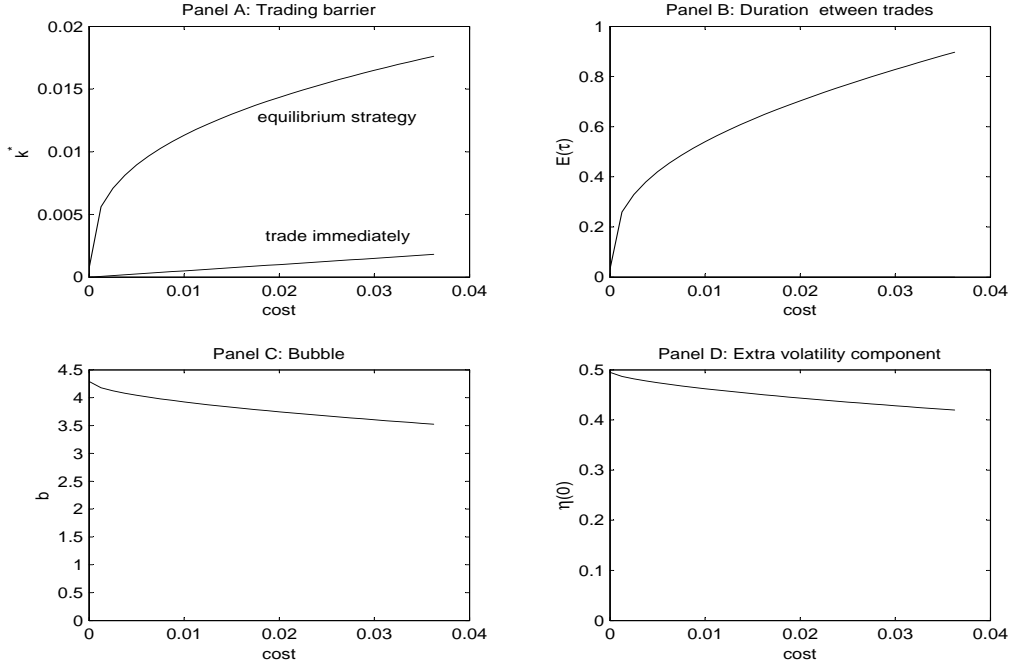


Figure 3: Effects of trading costs. The following parameters have been specified: $r = 5\%$, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$. The trading barrier, the bubble, the extra volatility component, and trading cost are all measured as multiples of $\frac{\sigma_f}{r+\lambda}$, the fundamental volatility of the asset.

8 Effects of trading costs

Using the results established in subsection 6.1, we can show that increasing the trading cost c raises the trading barrier k^* , and reduces b , $q(x)$ and $\eta(x)$. In fact:

Proposition 4 *If c increases, the optimal trading barrier k^* increases. Furthermore, the bubble b , the option component $q(x)$ and the excess volatility $\eta(x)$ ($\forall x \leq k^*(c)$) decrease. As $c \rightarrow 0$, $\frac{dk^*}{dc} \rightarrow \infty$, but the derivatives of b , $q(x)$, and $\eta(x)$ are always finite.*

Proof: See appendix.

In order to illustrate the effects of trading costs, we use the following parameter values from our previous numerical exercise, $r = 5\%$, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$. Figure 3 shows the effect of trading costs on the trading barrier k^* , expected duration between trades, the bubble b , and $\eta(0)$ (the extra volatility component when beliefs coincide).

Panel A of Figure 3 shows the equilibrium trading barrier k^* . For comparison, we also graph the amount $c(r + \lambda)$, which corresponds to the difference in beliefs that would justify trade if the option value was ignored. The difference between these two quantities represents the “profits” that the asset owner thinks he is obtaining when he exercises the option to sell. When the trading cost is zero, the asset owner sells the asset immediately when it is profitable and these profits are infinitely small. As the trading cost increases, the optimal trading barrier increases, and the rate of increase near $c = 0$ is dramatic, since the derivative $\frac{dk^*}{dc}$ is infinite at the origin. As a result, the trading frequency is greatly reduced by the increasing trading cost as shown in Panel B.

Panels C and D show that trading costs also reduce the bubble and the extra volatility component, but as guaranteed by Proposition 4, at a limited rate even near $c = 0$. Although one could expect that the strong reduction in trading frequency should greatly reduce the bubble, this effect is partially offset by the increase in profits in each trade.²⁰ Similar intuition applies to the effect of the trading cost on the extra volatility component.

To estimate the impact of an increase on trading costs, measured as a proportion of price, as opposed to a multiple of fundamental volatility, we must take a stand concerning the relationship between price and volatility of fundamentals. For the parameter values used in our example, Panel C shows that the bubble, for $c \sim 0$, is close to four times the fundamental volatility parameter $\frac{\sigma f}{r + \lambda}$. If we accept that in the case of the internet bubble, the size of the bubble in the late 90’s was of the order of 80% of the price that prevailed in the late 90’s, by comparing average prices in the late 90’s with today’s prices, then the size of the fundamental volatility is of the same magnitude as 20% of trading prices, and we can reinterpret the values in the figures as multiples of prices. The numerical results indicate that in this case a tax of 1% of prices would have caused a reduction of less than 20% to the magnitudes of both the bubble and the extra volatility component.

The effectiveness of a trading tax in reducing speculative trading has been hotly debated since James Tobin’s (1978) initial proposal for a transaction tax in the foreign currency markets. Shiller (2000, pages 225-228) provides an overview of the current

²⁰Vayanos (1998) makes a similar point in a different context, when analyzing the effects of transaction cost on asset prices in a life-cycle model. Vayanos shows that an increase of transaction cost can reduce the trading frequency but may even increase asset prices.

status of this debate. Our model implies that for small trading costs, increases in trading costs have a much larger impact in trading frequency than in excess volatility or the magnitude of the price bubble. In reality, trading also occurs for other reasons, such as liquidity shocks or changes in risk bearing capacity, that are not considered in our analysis and, for this reason, the limited impact of transaction costs on volatility and price bubbles cannot serve as an endorsement of a tax on trading. Our numerical exercise can also answer a question raised by Shiller (2000) of why bubbles can exist in real estate markets, where the transaction costs are typically high.

9 Can the price of a subsidiary be larger than its parent firm?

The existence of the option value component in asset prices can potentially create violations to the law of one price and even make the price of a subsidiary exceed that of a parent company. In this section, we use an example to illustrate this type of situation.

There are two firms, indexed by 1 and 2. For simplicity, we assume the dividend processes of both assets follow the process in equation (1) with the same parameter σ_D , but with independent innovations, and with different fundamental variables f_1 and f_2 respectively. The fundamental variables f_1 and f_2 are unobservable and both follow the linear mean-reverting process in equation (2) with the same parameters λ , \bar{f} and σ_f . To illustrate our point, we consider a special case in which the innovations in the processes of f_1 and f_2 are perfectly negatively correlated and there are no trading costs.

There is a third firm, and the dividend flow of firm 3 is exactly the sum of the dividend flows of firms 1 and 2. In this sense, firms 1 and 2 are both subsidiaries of firm 3, and the fundamental variable of firm 3 is the sum of that of firms 1 and 2: $f_3 = f_1 + f_2$. Since the innovations of f_1 and f_2 are perfectly negatively correlated, f_3 is a constant determined by initial conditions.

Shares of these three firms are traded by the two groups of agents described in Section 2. Since the fundamental variables of firms 1 and 2 fluctuate and are unobservable, these agents try to infer their values. According to our earlier discussion, overconfidence generates heterogeneous beliefs among agents in different groups. As a result, an option

component exists in the prices of the shares of firm 1 and firm 2. Since innovations to the fundamental variables f_1 and f_2 are perfectly negatively correlated, the beliefs of agents about these two assets are also perfectly negatively correlated, i.e., when \hat{f}_1^A (\hat{f}_1^B) moves up by certain amount, \hat{f}_2^A (\hat{f}_2^B) moves down by the same amount. Since $c = 0$, agents with higher beliefs hold the asset. Therefore, when agents in group A are holding firm 1, agents in group B must be holding firm 2, and the option components in the prices of these two firms are always the same. Therefore, the prices of firms 1 and 2 can be expressed as

$$p_1 = \frac{\bar{f}}{r} + \frac{\hat{f}_1 - \bar{f}}{r + \lambda} + q(x), \quad p_2 = \frac{\bar{f}}{r} + \frac{\hat{f}_2 - \bar{f}}{r + \lambda} + q(x), \quad (55)$$

where $x = \hat{f}_1^{\bar{o}} - \hat{f}_1^{\circ} = \hat{f}_2^{\bar{o}} - \hat{f}_2^{\circ} < 0$.

Since agents in both groups know that the fundamental variable of firm 3 is a constant, there are no heterogeneous beliefs about f_3 . Therefore, there is no option component or bubble in the price of firm 3. The price of firm 3 can be expressed as

$$p_3 = \frac{2\bar{f}}{r} + \frac{\hat{f}_1 + \hat{f}_2 - 2\bar{f}}{r + \lambda}. \quad (56)$$

According to the numerical exercise in Subsection 7.4, the magnitude of the option component in the prices of assets 1 and 2 can equal four or five times their fundamental volatility. If fundamental volatility is large relative to the discounted value of fundamentals, the value of one of the subsidiaries will exceed the value of firm 3, even though all prices are nonnegative.²¹ Although highly stylized, this analysis may help clarify the episodes such as 3Com's equity carve-out of Palm and its subsequent spinoff.²² In early 2000, for a period of over two months the total market capitalization of 3Com was significantly less than the market value of its holding in Palm, a subsidiary of 3Com. Similar situations also happened in other carve-out cases studied in Lamont and Thaler (2001), Mitchell, Pulvino and Stafford (2001), and Schill and Zhou (2000). In this example, our model also predicts that trading in the subsidiary would be much higher than trading in

²¹Duffie, Garleanu, and Pedersen (2001) provide another mechanism to explain this phenomenon based on the lending fee that the asset owner can expect to collect.

²²The missing link is to demonstrate that the divergence of beliefs on the combined entity was smaller than the divergence of beliefs on the Palm spinoff.

the parent company, because of the higher fluctuation in beliefs about the value of the subsidiary. In fact, Lamont and Thaler (2001) remarked that the turnover rate of the subsidiaries' stocks was on average six times higher than that of the parent firms' stocks.

This example also illustrates the fact that the diversification of a firm reduces the bubble component in the firm's stock price because diversification reduces the fundamental uncertainty of the firm and therefore reducing the potential disagreements among investors. This result is consistent with the diversification discount "puzzle" - the fact that the stocks of diversified firms appear to trade at a discount compared to the stocks of undiversified firms.²³

10 Conclusion and further discussions

In this paper, we provide a simple model to study bubbles and trading volume that result from speculative trading among agents with heterogeneous beliefs. Heterogeneous beliefs arise from the presence of overconfident agents. With a short-sale constraint, an asset owner has an option to sell the asset to other agents with more optimistic beliefs. Agents value this option, and consequently pay prices that exceed their own valuation of future dividends, because they believe that in the future they will find a buyer willing to pay even more. We solve the optimal exercise problem of an asset owner and write down, in an almost analytic form, many of the equilibrium variables of interest. This allows us to characterize properties of the magnitude of the bubble, trading frequency, and asset return volatility and to show that the model is consistent with observations that have been made about actual historical bubbles. Theoretical results and numerical simulations suggest that a small trading tax may be effective in reducing speculative trading, but it may not be very effective in reducing price volatility or the size of the bubble. Through a simple example, we also illustrate that the bubble can cause the price of a subsidiary to be larger than its parent firm, a violation of the law of one price.

It is natural to speculate that the existence of a speculative component in asset prices has implications for corporate strategies. Firm managers may be able to profit by

²³See Lang and Stultz (1994), Burger and Ofek (1995), and others.

adopting strategies that boost the speculative component.

The underpricing of a firm's initial public offering (IPO) has been puzzling. As reviewed by Ritter (2002), the average first day return of an IPO is about 10 to 15 percent. For the recent internet stock IPOs, it was common to see first day returns of 50% or even more than 100%. In some cases hundreds of millions of dollars were left on the table. Rajan and Servaes (1997), and Aggarwal, Krigman, and Womack (2001) show that higher initial returns on an IPO lead to more analysts and media coverage. Since investors may disagree about the precision of information provided by analysts and media, the increase in this coverage could increase the option component of the stock. Therefore, IPO underpricing could be a strategy used by firm managers to boost the price of their stocks. Firm managers, who typically hold residual shares, can get greater payoffs from subsequent sales of their own personal shares after the lock-up period. If this mechanism is operative, underpricing is more likely to occur when managers hold a larger share of the firm. This agrees with the empirical results in Aggarwal, Krigman, and Womack (2001) who show that managerial share and option holdings are positively related to first day IPO underpricing. If underpricing occurs because of the mechanism we propose, a larger underpricing should be associated with a larger trading volume. In fact, Reese (2000) finds that the higher initial IPO returns is associated with larger trading volume for more than three years after issuance.

Another popular corporate strategy during the recent internet stock bubble was to change firm name to a "dotcom" name. Cooper, Dimitrov, and Rau (2001) use a sample of 147 firms that changed their names to a dotcom name between June 1998 and July 1999, to document abnormal returns on the order of 53 percent in the five days around the announcement date. Lee (2001) also documents that the average trading volume rises twelve fold on the announcement date in a sample of 114 firms that change their names to dotcom between January 1995 and June 1999, even though these name changes were not accompanied by any changes in strategy. If as it seems likely the name change increased the attention of the analyst, and if investors disagree on the precision of information provided by analysts and media, the name change would increase the speculative component of price.

Since stocks have been widely used in compensation contracts for firm executives, the presence of bubbles in stock prices will lead to different managerial incentives than those discussed in standard theories of executive compensation that assume unbiased stock prices. In Bolton, Scheinkman, and Xiong (2002), we formally analyze managerial contracts in a model of speculative markets that is based in the framework presented in this paper. We show that the presence of overconfidence on the part of potential stock buyers could induce incumbent shareholders to use short-term stock compensation to motivate managerial behavior that increases short term prices at the expense of long term performance. This provides an alternative to the common view that the recent corporate scandals were caused by a lack of adequate board supervision.

A Proofs

A.1 Proof to Lemma 1

Let $\vartheta(\phi) = \lambda + \phi \frac{\sigma_f}{\sigma_s}$ and $\iota(\phi) = (1 - \phi^2)(2\sigma_f^2/\sigma_s^2 + \sigma_f^2/\sigma_D^2)$. Then

$$\frac{d\gamma}{d\phi} \sim \frac{1}{2} \frac{2\vartheta \frac{d\vartheta}{d\phi} + \frac{d\iota}{d\phi}}{\sqrt{(\vartheta^2 + \iota)}} - \frac{d\vartheta}{d\phi} = \left(\frac{\vartheta}{\sqrt{\vartheta^2 + \iota}} - 1 \right) \frac{d\vartheta}{d\phi} + \frac{1}{2\sqrt{\vartheta^2 + \iota}} \frac{d\iota}{d\phi} \leq 0. \quad (\text{A1})$$

A.2 Proof to Proposition 1

The process of g^A can be derived from the conditional beliefs \hat{f}^A and \hat{f}^B in equations (8) and (12):

$$dg^A = d\hat{f}^B - d\hat{f}^A = - \left[\lambda + \frac{2\gamma + \phi\sigma_s\sigma_f}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2} \right] g^A dt + \frac{\phi\sigma_f}{\sigma_s} (ds^B - ds^A). \quad (\text{A2})$$

The difference of beliefs g^A mean-reverts with a parameter of

$$\begin{aligned} \rho &= \lambda + \frac{2\gamma + \phi\sigma_s\sigma_f}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2} \\ &= \sqrt{\left(\lambda + \phi \frac{\sigma_f}{\sigma_s} \right)^2 + (1 - \phi^2)\sigma_f^2 \left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2} \right)}. \end{aligned} \quad (\text{A3})$$

In the mind of agents in group A,

$$ds^A = \hat{f}_A dt + \sigma_s dW_A^A, \quad (\text{A4})$$

$$ds^B = \hat{f}_A dt + \sigma_s dW_B^A, \quad (\text{A5})$$

according to equations (9) and (10). Therefore,

$$dg^A = -\rho g^A dt + \frac{\phi\sigma_f}{\sigma_s} (\sigma_s dW_B^A - \sigma_s dW_A^A). \quad (\text{A6})$$

We can simplify the notation to

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A \quad (\text{A7})$$

with

$$\sigma_g = \sqrt{2}\phi\sigma_f, \quad (\text{A8})$$

$$dW_g^A = \frac{1}{\sqrt{2}} (dW_B^A - dW_A^A). \quad (\text{A9})$$

It is easy to verify that W_g^A is independent to the innovations to \hat{f}^A in the mind of agents in group A.

Similar derivation can be done for the difference of beliefs g^B in the mind of agents in group B.

A.3 Proof to Proposition 2

Let $v(y)$ be a solution to the differential equation

$$yv''(y) + (1/2 - y)v'(y) - \frac{r}{2\rho}v(y) = 0. \quad (\text{A10})$$

It is straightforward to verify that

$$u(x) = v\left(\frac{\rho}{\sigma_g^2}x^2\right) \quad (\text{A11})$$

satisfies the equation :

$$\frac{1}{2}\sigma_g^2u''(x) - \rho xu'(x) = ru(x). \quad (\text{A12})$$

The general solution of equation (A10) is²⁴

$$v(y) = \alpha M\left(\frac{r}{2\rho}, \frac{1}{2}, y\right) + \beta U\left(\frac{r}{2\rho}, \frac{1}{2}, y\right). \quad (\text{A13})$$

$M(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot, \cdot)$ are Kummer functions defined as

$$M(a, b, y) = 1 + \frac{ay}{b} + \frac{(a)_2y^2}{(b)_22!} + \dots + \frac{(a)_ny^n}{(b)_nn!} + \dots \quad (\text{A14})$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \quad (a)_0 = 1, \quad (\text{A15})$$

and

$$U(a, b, y) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, y)}{\Gamma(1+a-b)\Gamma(b)} - y^{1-b} \frac{M(1+a-b, 2-b, y)}{\Gamma(a)\Gamma(2-b)} \right\}. \quad (\text{A16})$$

²⁴See Abramowitz and Stegun (1964), chapter 13.

In addition:

$$M_y(a, b, y) > 0, \quad \forall y > 0 \quad (\text{A17})$$

$$M(a, b, y) \rightarrow +\infty, \quad U(a, b, y) \rightarrow 0, \quad \text{as } y \rightarrow +\infty. \quad (\text{A18})$$

Given a solution u to equation (A12) we can construct two solutions v to equation (A10), by using the values of the function for $x < 0$ and for $x > 0$. We will denote the corresponding linear combinations of M and U by $\alpha_1 M + \beta_1 U$ and $\alpha_2 M + \beta_2 U$. If these combinations are constructed from the same u their values and first derivatives (and consequently second derivatives) must have the same limit as $x \rightarrow 0$. To guarantee that $u(x)$ is positive and increasing for $x < 0$, α_1 must be zero. Therefore,

$$u(x) = \beta_1 U \left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2} x^2 \right) \quad \text{if } x \leq 0. \quad (\text{A19})$$

The solution must be continuously differentiable at $x = 0$. From the definition of the two Kummer functions, we can show that

$$\begin{aligned} x \rightarrow 0-, \quad u(x) &\rightarrow \frac{\beta_1 \pi}{\Gamma(\frac{1}{2} + \frac{r}{2\rho}) \Gamma(\frac{1}{2})}, & u'(x) &\rightarrow \frac{\beta_1 \pi \sqrt{\rho}}{\sigma_g \Gamma(\frac{r}{2\rho}) \Gamma(\frac{3}{2})} \\ x \rightarrow 0+, \quad u(x) &\rightarrow \alpha_2 + \frac{\beta_2 \pi}{\Gamma(\frac{1}{2} + \frac{r}{2\rho}) \Gamma(\frac{1}{2})}, & u'(x) &\rightarrow -\frac{\beta_2 \pi \sqrt{\rho}}{\sigma_g \Gamma(\frac{r}{2\rho}) \Gamma(\frac{3}{2})} \end{aligned} \quad (\text{A20})$$

By matching the values and first order derivatives of $u(x)$ from the two sides of $x = 0$, we have

$$\beta_2 = -\beta_1, \quad \alpha_2 = \frac{2\beta_1 \pi}{\Gamma(\frac{1}{2} + \frac{r}{2\rho}) \Gamma(\frac{1}{2})}. \quad (\text{A21})$$

The function h is obviously a solution to equation (A12) that is twice differentiable and satisfies

$$h(0) = \frac{\pi}{\Gamma(\frac{1}{2} + \frac{r}{2\rho}) \Gamma(\frac{1}{2})} > 0, \quad (\text{A22})$$

and $h(-\infty) = 0$. Equation (A12) guarantees that at any critical point where $h < 0$, h has a maximum, and at any critical point where $h > 0$ it has a minimum. Hence h is strictly positive and increasing in $(-\infty, 0)$.

A.4 Proof to Lemma 2

$h(x)$ is solution to the

$$\alpha h'' - xh' - \beta h = 0, \quad (\text{A23})$$

where $\alpha = \frac{\sigma_g^2}{2\rho} > 0$ and $\beta = \frac{r}{\rho} > 0$, that is positive and increasing in $(-\infty, 0)$.

If $x^* \in R$ with $h(x^*) > 0$ and $h'(x^*) = 0$ then $h''(x^*) = \beta h(x^*)/\alpha > 0$. Hence h has no local maximum while it is positive and as a consequence it is always positive and has no local maxima. In particular h is monotonically increasing. Since $h' > 0$ for $x \leq 0$ and $h'' \geq 0$ for $x \geq 0$, $h'(x) > 0$ for all x . From the solution constructed in Proposition 2, $\lim_{x \rightarrow -\infty} h(x) = 0$.

Note any solution to the differential equation is infinitely differentiable. Next, we show that h is convex. For $x > 0$, $h''(x) = xh'(x)/\alpha + \beta h(x)/\alpha > 0$. To prove that h is also convex for $x < 0$, let us assume that there exists $x^* < 0$ such that $h''(x^*) \leq 0$. Then

$$h'''(x^*) = x^*h''(x^*)/\alpha + (\beta + 1)h'(x^*)/\alpha > 0. \quad (\text{A24})$$

This directly implies that $h''(x) < 0$ for $x < x^*$. Then $\lim_{x \rightarrow -\infty} h'(x) = \infty$. In this situation the boundary condition $h(-\infty) = 0$ can not be satisfied. In this way, we get a contradiction.

Let $v(x) = h'(x)$. $v(x)$ is positive and increasing from the properties that we have proved for $h(x)$. v also satisfies the following equation:

$$\alpha v''(x) - xv'(x) - (\beta + 1)v(x) = 0. \quad (\text{A25})$$

This equation is very similar to the one satisfied by $h(x)$. By repeating the same proof for h , one can show that $v(x)$ is also convex and $\lim_{x \rightarrow -\infty} v(x) = 0$.

Actually, one can show that any higher order derivative of $h(x)$ is positive, increasing and convex.

A.5 Proof to Theorem 1

Let

$$l(k) = [k - c(r + \lambda)](h'(k) + h'(-k)) - h(k) + h(-k). \quad (\text{A26})$$

We first show that there exists a unique k^* that solves $l(k) = 0$.

If $c = 0$, $l(0) = 0$, and

$$l'(k) = k[h''(k) - h''(-k)] > 0, \text{ for all } k \neq 0. \quad (\text{A27})$$

Therefore $k^* = 0$ is the only root to $l(k) = 0$.

If $c > 0$, then

$$l(k) < 0, \text{ for all } k \in [0, c(r + \lambda)]. \quad (\text{A28})$$

Also, since h'' and h''' are increasing (Lemma 2),

$$l'(k) = [k - c(r + \lambda)][h''(k) - h''(-k)] > 0, \quad \forall k > c(r + \lambda), \quad (\text{A29})$$

$$l''(k) = h''(k) - h''(-k) + [k - c(r + \lambda)][h'''(k) - h'''(-k)] > 0, \quad \forall k > c(r + \lambda). \quad (\text{A30})$$

Therefore $l(k) = 0$ has a unique solution $k^* > c(r + \lambda)$.

A.6 Proof to Theorem 2

First we show that q satisfies equation (28). Using equation (39), we have

$$q(-x) = \begin{cases} \frac{b}{h(-k^*)}h(-x) & \text{for } x > -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}h(x) - c & \text{for } x \leq -k^*. \end{cases} \quad (\text{A31})$$

We must establish that

$$U(x) = q(x) - \frac{x}{r + \lambda} - q(-x) + c \geq 0, \quad \forall x. \quad (\text{A32})$$

A simple calculation shows that

$$U(x) = \begin{cases} 2c & \text{for } x < -k^* \\ \frac{-x}{r+\lambda} + \frac{b}{h(-k^*)}[h(x) - h(-x)] + c & \text{for } -k^* \leq x \leq k^* \\ 0 & \text{for } x > k^* \end{cases} \quad (\text{A33})$$

Thus,

$$U''(x) = \frac{b}{h(-k^*)}[h''(x) - h''(-x)], \quad -k^* \leq x \leq k^*. \quad (\text{A34})$$

From lemma 2 we know for $U''(x) > 0$ for $0 < x < k^*$, and $U''(x) < 0$ for $-k^* < x < 0$. Since $U'(k^*) = 0$, $U'(x) < 0$ for $0 < x < k^*$. On the other hand, $U'(-k^*) = 0$, so $U'(x) <$

0 for $-k^* < x < 0$. Therefore $U(x)$ is monotonically decreasing for $-k^* < x < k^*$. Since $U(-k^*) = 2c > 0$ and $U(k^*) = 0$, $U(x) > 0$ for $-k^* < x < k^*$.

We now show that equation (29) holds. By construction, equation (29) holds in the region $x \leq k^*$. Therefore we only need to show for $x \geq k^*$,

$$\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - r q(x) \leq 0. \quad (\text{A35})$$

In this region, $q(x) = \frac{x}{r+\lambda} + \frac{b}{h(-k^*)}h(-x) - c$, thus $q'(x) = \frac{1}{r+\lambda} - \frac{b}{h(-k^*)}h'(-x)$ and $q''(x) = \frac{b}{h(-k^*)}h''(-x)$. Hence,

$$\begin{aligned} & \frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - r q(x) \\ &= \frac{b}{h(-k^*)} \left[\frac{1}{2}\sigma_g^2 h''(-x) + \rho x h'(-x) - r h(-x) \right] - \frac{r+\rho}{r+\lambda}x + rc \\ &= -\frac{r+\rho}{r+\lambda}x + rc \leq -(r+\rho)c + rc = -\rho c < 0 \end{aligned} \quad (\text{A36})$$

where the inequality comes from the fact that $x \geq k^* > (r+\lambda)c$ from Theorem 1.

Also q has an increasing derivative in $(-\infty, k^*)$ and has a derivative bounded in absolute value by $\frac{1}{r+\lambda}$ in (k^*, ∞) . Hence q' is bounded.

If τ is any stopping time, the version of Ito's lemma for twice differentiable functions with absolutely continuous first derivatives (*e.g.* Revuz and Yor (1999), Chapter VI) implies that

$$e^{-r\tau} q(g_\tau^o) = q(g_0^o) + \int_0^\tau \left[\frac{1}{2}\sigma_g^2 q''(g_s^o) - \rho g_s^o q'(g_s^o) - r q(g_s^o) \right] ds + \int_0^\tau \sigma_g q'(g_s^o) dW_s \quad (\text{A37})$$

Equation (29) states that the first integral is non positive, while the bound on q' guarantees that the second integral is a Martingale. Using equation (28) we obtain,

$$\mathbb{E}^o \left\{ e^{-r\tau} \left[\frac{g_\tau^o}{r+\lambda} + q(-g_\tau^o) - c \right] \right\} \leq \mathbb{E}^o \left[e^{-r\tau} q(g_\tau^o) \right] \leq q(g_0^o). \quad (\text{A38})$$

This shows that no policy can yield more than $q(x)$.

Now consider the stopping time $\tau = \inf\{t : g_t^o \geq k^*\}$. Such τ is finite with probability one, and g_s^o is in the continuation region for $0 \leq s < \tau$. Hence using exactly the same reasoning as above, but recalling that in the continuation region (29) holds with equality we obtain that

$$q(g^o) = \mathbb{E}^o \left\{ e^{-r\tau} \left[\frac{g_\tau^o}{r+\lambda} + q(-g_\tau^o) - c \right] \right\}. \quad (\text{A39})$$

A.7 Proof to Proposition 3

Since $q(x) = \frac{1}{(r+\lambda)} \frac{h(x)}{h'(k)+h'(-k)}$, the volatility of $q(g_t^o)$ is given by $\frac{1}{(r+\lambda)} \frac{h'(g_t^o)}{h'(k)+h'(-k)}$ multiplied by the volatility of g_t^o . From the proof to proposition 1,

$$dg_t^o = -\rho g_t^o dt + \frac{\phi \sigma_f}{\sigma_s} (ds^{\bar{o}} - ds^o). \quad (\text{A40})$$

We need to determine the volatility of this process from the perspective of an objective econometrician. From equations (3) and (4) the volatility of $s^{\bar{o}} - s^o$ is $\sqrt{2}\sigma_s$ in an objective measure. Hence the volatility of g^o is $\sqrt{2}\phi\sigma_f$.

A.8 Proof to Lemma 3

When $c = 0$, the magnitude of the bubble is

$$b_0 = \frac{\sigma_g}{2\sqrt{2\rho}(r+\lambda)} \frac{\Gamma\left(\frac{r+\theta}{2\rho}\right)}{\Gamma\left(\frac{1}{2} + \frac{r+\theta}{2\rho}\right)}. \quad (\text{A41})$$

It is obvious that b_0 increases with σ_g . We can directly show that b_0 increases with ρ and decreases with r and θ by plotting it.

When $c = 0$, the option value component is $q_0(x) = b_0 \frac{h(x)}{h(0)}$ where $h(x)$ is a positive and increasing solution to

$$\frac{1}{2}\sigma_g^2 h''(x) - \rho x h'(x) - (r+\theta)h(x) = 0, \quad h(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r+\theta}{2\rho}\right) \Gamma\left(\frac{1}{2}\right)}. \quad (\text{A42})$$

Note that $q_0(x)$ is not effected by letting $h(0) = 1$.

Assume $\tilde{\sigma}_g > \sigma_g$, let $\tilde{h}(x)$ satisfy the following differential equation

$$\frac{1}{2}\tilde{\sigma}_g^2 \tilde{h}''(x) - \rho x \tilde{h}'(x) - (r+\theta)\tilde{h}(x) = 0, \quad \tilde{h}(-\infty) = 0, \quad \tilde{h}(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r+\theta}{2\rho}\right) \Gamma\left(\frac{1}{2}\right)}. \quad (\text{A43})$$

We can show $\tilde{h}(x) > h(x)$ for all $x < 0$. Let

$$f(x) = \tilde{h}(x) - h(x). \quad (\text{A44})$$

Then $f(-\infty) = f(0) = 0$ using Lemma 2. $f(x)$ has no local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, from the equations satisfied by $\tilde{h}(x)$ and $h(x)$, we have

$$\frac{1}{2}[\tilde{\sigma}_g^2 \tilde{h}''(x) - \sigma_g^2 h''(x)] - \rho x[\tilde{h}'(x) - h'(x)] - (r+\theta)[\tilde{h}(x) - h(x)] = 0. \quad (\text{A45})$$

This equation implies that

$$\tilde{\sigma}_g^2 \tilde{h}''(x^*) < \sigma_g^2 h''(x^*). \quad (\text{A46})$$

Since $\tilde{\sigma}_g^2 > \sigma_g^2$, this further implies that $\tilde{h}''(x^*) < h''(x^*)$. This is equivalent to $f''(x^*) < 0$, which contradicts with x^* being a local minimum. Therefore, $f(x)$ cannot have a local minimum with its value less than zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x \in (-\infty, 0)$. Therefore, $\tilde{h}(x) > h(x)$ for all $x < 0$. This directly implies that the option value component $q_0(x)$ increases with σ_g for all $x < 0$.

Assume $\bar{\rho} > \rho$, let $\bar{h}(x)$ satisfy the following differential equation

$$\frac{1}{2}\sigma_g^2 \bar{h}''(x) - \bar{\rho}x \bar{h}'(x) - (r + \theta)\bar{h}(x) = 0, \quad \bar{h}(-\infty) = 0, \quad \bar{h}(0) = \frac{\pi}{\Gamma\left(\frac{1}{2} + \frac{r+\theta}{2\rho}\right) \Gamma\left(\frac{1}{2}\right)}. \quad (\text{A47})$$

We can show $\bar{h}(x) < h(x)$ for all $x < 0$. Again let

$$f(x) = \bar{h}(x) - h(x). \quad (\text{A48})$$

We first establish that $f(x)$ has no local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, from the equations satisfied by $\bar{h}(x)$ and $h(x)$, we have

$$\frac{1}{2}\sigma_g^2[\bar{h}''(x) - h''(x)] - \rho x[\bar{h}'(x) - h'(x)] - (r + \theta)[\bar{h}(x) - h(x)] = (\bar{\rho} - \rho)x\bar{h}'(x). \quad (\text{A49})$$

This equation implies that

$$\bar{h}'(x^*) < 0, \quad (\text{A50})$$

which contradicts with $\bar{h}(x)$ as an increasing function. Therefore, $f(x)$ cannot have a local minimum below zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x < 0$. This directly implies that $\bar{h}(x) > h(x)$ for all $x < 0$, and $q_0(x)$ increases with ρ for all $x < 0$. Similarly, we can prove that $q_0(x)$ decreases with r and θ for all $x < 0$.

One can extend the comparative statics we established for $c = 0$ for the case of c small. Let $\zeta \in \{\sigma_g, \rho, \theta\}$. From equation (38) it follows that if $\frac{\partial k^*(\zeta, c)}{\partial \zeta} = o(k^*)$ then the comparative statics of b with respect to ζ is preserved for small c .

Using the definition of function h in equation (31), we write h as $h(x, \zeta)$. From equation (37),

$$\frac{\partial k^*(\zeta, c)}{\partial \zeta} = - \frac{[k^* - c(r + \lambda)] \left(\frac{\partial^2 h(k^*, \zeta)}{\partial x \partial \zeta} + \frac{\partial^2 h(-k^*, \zeta)}{\partial x \partial \zeta} \right) - \left(\frac{\partial h(k^*, \zeta)}{\partial \zeta} - \frac{\partial h(-k^*, \zeta)}{\partial \zeta} \right)}{[k^* - c(r + \lambda)] \left[\frac{\partial^2 h(k^*, \zeta)}{\partial x^2} - \frac{\partial^2 h(-k^*, \zeta)}{\partial x^2} \right]}. \quad (\text{A51})$$

As $c \rightarrow 0$, $k^* \rightarrow 0$ and hence both the numerator and denominator go to zero. To find the limit behavior, we use the explicit form of h given in the proof of Proposition 2, and write

$$h(x, \zeta) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + o(x^4) \quad (\text{A52})$$

with

$$C_0 = \frac{\pi}{\Gamma\left(\frac{r}{2\rho} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \quad (\text{A53})$$

$$C_1 = \frac{\pi \sqrt{\rho}}{\Gamma\left(\frac{r}{2\rho}\right) \Gamma\left(\frac{3}{2}\right) \sigma_g} \quad (\text{A54})$$

$$C_2 = \frac{\pi r}{4\Gamma\left(\frac{r}{2\rho} + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \sigma_g^2} \quad (\text{A55})$$

$$C_3 = \frac{\pi \sqrt{\rho}(r + \rho)}{3\Gamma\left(\frac{r}{2\rho}\right) \Gamma\left(\frac{3}{2}\right) \sigma_g^3} \quad (\text{A56})$$

We can use equation (37) to replace the term $k^* - c(r + \lambda)$ in the right hand side of equation (A51) by $\frac{h(k^*, \zeta) - h(-k^*, \zeta)}{\frac{\partial h(k^*, \zeta)}{\partial x} + \frac{\partial h(-k^*, \zeta)}{\partial x}}$. Taking limits as $k^* \rightarrow 0$ we obtain,

$$\frac{\partial k^*(\zeta, c)}{\partial \zeta} \sim o(k^*), \quad (\text{A57})$$

$\zeta \in \{\sigma_g, \rho, \theta\}$. A small variation establishes the same result for $\frac{\partial k^*(r, c)}{\partial r}$. Hence, for small c , b increases with σ_g and ρ , and decrease with r and θ . In addition we can show that $q(x)$ for $x < k^*$ increases with σ_g and ρ , and decreases with r and θ .

A.9 Proof to Proposition 4

Let

$$l(k, c) = [k - c(r + \lambda)](h'(k) + h'(-k)) - h(k) + h(-k). \quad (\text{A58})$$

$k^*(c)$ is the root of $l(k, c) = 0$. If $c > 0$

$$\frac{dk^*}{dc} = \frac{(r + \lambda)}{[k^* - c(r + \lambda)]} \frac{[h'(k^*) + h'(-k^*)]}{[h''(k^*) - h''(-k^*)]} > 0. \quad (\text{A59})$$

Hence $k^*(c)$ is differentiable in $(0, \infty)$. Now suppose $c_n \rightarrow 0$. The sequence $k^*(c_n)$ is bounded and every limit point \bar{k}^* must solve $l(\bar{k}^*, 0) = 0$. Hence, as we argued in the proof in the proof of Theorem 1, $\bar{k}^* = 0$ and the function $k^*(c)$ is continuous. Hence $\frac{dk^*}{dc}$ approaches ∞ as $c \rightarrow 0$. The claims on b and $q(x)$ follow from equations (38) and (39), and Lemma 2. The derivative of $\eta(x)$ with respect to c is

$$\begin{aligned} \frac{d\eta(x)}{dc} &= \frac{\sqrt{2}\phi\sigma_f}{(r + \lambda)} \frac{h'(x)(h''(k^*) - h''(-k^*))}{(h'(k^*) + h'(-k^*))^2} \left(-\frac{dk^*}{dc}\right) \\ &= -\frac{\sqrt{2}\phi\sigma_f h'(x)}{[k^* - c(r + \lambda)](h'(k^*) + h'(-k^*))} < 0. \end{aligned} \quad (\text{A60})$$

Therefore, $\eta(x)$ decreases with c . However, note that $\frac{d\eta(x)}{dc}$ is finite as $c \rightarrow 0$ although $\frac{dk^*}{dc} \rightarrow \infty$ as $c \rightarrow 0$.

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