GENERALIZED LEAST SQUARES

– A solution to the following problem:

Suppose $Cov(\epsilon) \neq \sigma^2 I_T$

but rather $= \sigma^2 \Omega$, $\Omega \neq I_T$

i.e. $\epsilon \sim N(0, \sigma^2 \Omega)$

This violates the ideal conditions.
One or both of the following exists:

1. Heteroskedasticity
   - nonequal variances of the ϵ's ;
   - the diagonal elements of Ω are not necessarily equal ;
   - $\sigma_t^2 \neq \sigma_{t+1}^2$.

2. Autocorrelation or Serial Correlation
   - Ω is not necessarily a diagonal matrix ;
   - $Cov(\epsilon_t, \epsilon_{t+i}) \neq 0$;
   - disturbances not independent .

Properties of OLS :

Thm : $\hat{\beta} \sim N\left(\beta,\ \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}\right)$

Proof :

$$\hat{\beta} = \beta + (X'X)^{-1}X'\epsilon$$

i.e. $\hat{\beta}$ is linear in $\epsilon \rightarrow$ (i) l.c. of ϵ's ; $\therefore$ $\hat{\beta} \sim N$.

(ii) $E(\hat{\beta}) = \beta$; since $E(\epsilon) = 0$, $\hat{\beta}$ is unbiased.

(iii)
$$\begin{aligned}
Cov(\hat{\beta}) &= E[\hat{\beta}-\beta][\hat{\beta}-\beta]' \\
&= E[(X'X)^{-1}X'\epsilon][(X'X)^{-1}X'\epsilon]' \\
&= E(X'X)^{-1}X'\epsilon\epsilon'X(X'X)^{-1} \\
&= (X'X)^{-1}X'\,E(\epsilon\epsilon')\,X(X'X)^{-1} \\
&= \sigma^2 (X'X)^{-1}X'\Omega X(X'X)^{-1}
\end{aligned}$$

QED

Thm: $\hat{\beta}$ is consistent.

Proof :

$$\begin{aligned}
\text{plim}\ \hat{\beta} &= \text{plim}\left(\beta + (X'X)^{-1}X'\epsilon\right) \\
&= \beta + \text{plim}\ (X'X)^{-1}X'\epsilon \\
&= \beta + \underbrace{\underset{\Uparrow \text{ finite}}{\text{plim}\left(\frac{X'X}{T}\right)^{-1}}\ \underset{\Uparrow\ X\text{'s uncorrel. w/ } \epsilon}{\text{plim}\left(\frac{X'\epsilon}{T}\right)}}_{\text{Other ideal conditions still hold.}}
\end{aligned}$$

Alternative proof for consistency;

Since $\hat{\beta}$ unbiased, show that $Cov(\hat{\beta}) \to 0$.

$$Cov(\hat{\beta}) = \sigma^2 (X'X)^{-1} X'\Omega X (X'X)^{-1}$$

$$\text{plim } Cov(\hat{\beta}) = \lim_{T \to \infty} \sigma^2 \left(\frac{X'X}{T}\right)^{-1} \frac{X'\Omega X}{T^2} \left(\frac{X'X}{T}\right)^{-1}$$

$$= \sigma^2 \underbrace{\text{plim}\left(\frac{X'X}{T}\right)^{-1}}_{\text{finite}} \underbrace{\text{plim } \frac{X'\Omega X}{T^2}}_{0} \underbrace{\text{plim}\left(\frac{X'X}{T}\right)^{-1}}_{\text{finite}}$$

[for any reasonable Ω]

QED

$\therefore$ $\hat{\beta}$ unbiased & consistent.
But $\hat{\beta}$ is not efficient. GLS is better (later).

Thm: s^2 is biased and inconsistent.

Proof:
$$
\begin{aligned}
s^2 &= \frac{1}{T-K}\,\epsilon' M \epsilon \\
&= \frac{1}{T-K}\,\text{Trace}(\epsilon' M \epsilon) \qquad [\epsilon' M \epsilon \text{ a scalar}] \\
&= \frac{1}{T-K}\,\text{Tr}(M\epsilon\epsilon')
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E(s^2) &= \frac{1}{T-K}\,\text{Tr}(M\,E(\epsilon\epsilon')) \\
&= \frac{1}{T-K}\,\sigma^2\,\text{Tr}(M\Omega)
\end{aligned}
$$

$\neq \sigma^2$ unless $\text{Tr}(M\Omega) = T-K$,
which it isn't, unless $\Omega = I_T$.

Thus, s^2 is biased.
Pf. of inconsistency is similar.

Implications: The t & F tests are incorrect.

– for 2 reasons –
$$\frac{\hat{\beta}_i}{\sqrt{s^2 (X'X)^{-1}_{ii}}}$$

(i) s^2 is biased
(ii) $(X'X)^{-1}$ is not the appropriate covariance matrix
– $(X'X)^{-1} X'\Omega X (X'X)^{-1}$ is !

Summary:

① $\hat{\beta}_{OLS}$ unbiased, consistent, but inefficient.
② s^2 biased & inconsistent.
③ t & F tests are incorrect

Now, the Cure for $\epsilon \sim N(0, \sigma^2 \Omega)$

GLS - Generalized LS

Lemma: $\exists$ a nonsingular $T \times T$ matrix V $\ni$ $V'V = \Omega^{-1}$.

Proof: Ω is a covariance matrix, $\therefore$ psd.
Thus if Ω^{-1} exists, it is psd.
Thus $\exists$ a V $\ni$ $V'V = \Omega^{-1}$. (Schmidt p. 66)

Thm: Suppose $Y = X\beta + \epsilon$ satisfies ideal conditions except that $\text{Cov}(\epsilon) = \sigma^2 \Omega$.
Let $V'V = \Omega^{-1}$. Then

$(VY) = (VX)\beta + (V\epsilon)$
satisfies the ideal conditions.

Proof: (i) $V\epsilon$ is Normal.
(ii) $E(V\epsilon) = V E(\epsilon) = 0$.
(iii)
$$
\left.
\begin{aligned}
\text{Cov}(V\epsilon) &= E(V\epsilon\epsilon'V') \\
&= V(\sigma^2 \Omega)V' \\
&= \sigma^2 V\Omega V' \\
&= \sigma^2 I_T
\end{aligned}
\right\} \; V\epsilon \sim N(0, \sigma^2 I)
$$
(ideal conditions)

$V^T V \Omega V^T = V^T a$

$\cancel{V\Omega^{-1}}$

$V\Omega V^T = a$

Intuition: $V\Omega V' = I$ since if we premultiply by V', we get V'.

$V'(V\Omega V') = \Omega^{-1}\Omega V' = V'$

this $\Rightarrow$ $V\Omega V' = T$ since V' is nonsingular.

YUCK (see next page)

$$\mathrm{Cov}(V\epsilon) = E[(V\epsilon)(V\epsilon)']$$
$$= E[V\epsilon\epsilon' V]$$
$$= V.E[\epsilon\epsilon'].V'$$
$$= V.\sigma^2\Omega.V'$$
$$= \sigma^2 V\Omega V'$$

Now $V\Omega V' = V\Omega V'.I_T$

$$= V\Omega V'.VV^{-1}$$ (V^{-1} exists since V non-singular)
$$= V\Omega.V'V.V^{-1}$$
$$= V\Omega.\Omega^{-1}.V^{-1}$$
$$= V V^{-1}$$
$$= I_T$$

Hence $\mathrm{Cov}(V\epsilon) = \sigma^2 I_T$

Peter Gillett

Furthermore, VX is nonstochastic matrix of Rank K
$\ni \lim_{T\to\infty} \frac{(VX)'(VX)}{T}$ is finite & nonsingular.

$\Rightarrow$ given $\lim_{T\to\infty} \frac{X'X}{T}$ is finite,

$$\lim_{T\to\infty} \frac{X'V'VX}{T} = \lim_{T\to\infty} \frac{X'\Omega^{-1}X}{T} \quad \text{is finite}$$

for any reasonable Ω.

Thus the transformed equation, $(VY) = (VX)\beta + (V\epsilon)$, will have all the optimal properties under the ideal conditions.

<u>Defn:</u> The GLS estimator is the OLS estimator of $\underset{T\times 1}{(VY)}$ on $\underset{T\times K}{(VX)}$.

$$\begin{aligned} \tilde{\beta} &= [(VX)'(VX)]^{-1}(VX)'(VY) \\ &= (X'V'VX)^{-1}X'V'VY \\ &= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y \end{aligned}$$

$$\begin{aligned} \tilde{\sigma}^2 &= \frac{1}{T-K}[VY - VX\tilde{\beta}]'[VY - VX\tilde{\beta}] \\ &= \frac{1}{T-K}[Y - X\tilde{\beta}]'V'V[Y - X\tilde{\beta}] \\ &= \frac{1}{T-K}(Y - X\tilde{\beta})'\Omega^{-1}(Y - X\tilde{\beta}) \\ &= \frac{1}{T-K}\tilde{\epsilon}'\Omega^{-1}\tilde{\epsilon} \end{aligned}$$

This is a <u>weighted</u> sum of the squared residuals; weighted by Ω^{-1}.

<u>Fact:</u> GLS $\tilde{\beta}$ minimizes $SSE_{GLS} = (Y - X\beta)' \Omega^{-1} (Y - X\beta)$.

This is an algebraic result. — should be able to prove.

Proof: GLS - $\quad VY = VX\beta + V\epsilon$

$$\begin{aligned} \Longrightarrow SSE_{GLS} &= (VY - VX\beta)'(VY - VX\beta) \\ &= (Y - X\beta)' \Omega^{-1} (Y - X\beta) \\ &= Y'\Omega^{-1}Y - 2\beta' X' \Omega^{-1} Y + \beta' X' \Omega^{-1} X \beta \end{aligned}$$

$$\frac{\partial SSE_{GLS}}{\partial \beta} = -2X'\Omega^{-1}Y + 2X'\Omega^{-1}X\beta$$

$$\begin{aligned} \Longrightarrow \tilde{\beta} &= (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} Y \\ &= \beta + (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} \epsilon \end{aligned}$$

QED

<u>Consider the distribution of GLS $\tilde{\beta}$.</u>

<u>Thm:</u> $\tilde{\beta} \sim N(\beta, \sigma^2 (X'\Omega^{-1}X)^{-1})$

(i) $\tilde{\beta}$ l.c. of ϵ's ; $\therefore$ Normal

(ii) $E(\tilde{\beta}) = \beta$

(iii)
$$\begin{aligned} Cov(\tilde{\beta}) &= E(\tilde{\beta} - \beta)(\tilde{\beta} - \beta)' \\ &= E[(X'\Omega^{-1}X)^{-1} X'\Omega^{-1}\epsilon][\epsilon'\Omega^{-1}X(X'\Omega^{-1}X)^{-1}]^T \\ &= (X'\Omega^{-1}X)^{-1} X'\Omega^{-1} [\sigma^2 \Omega] \Omega^{-1} X (X'\Omega^{-1}X)^{-1} \\ &= \sigma^2 (X'\Omega^{-1}X)^{-1} \end{aligned}$$

QED

Thus, since the transformed equation satisfies the ideal conditions,

$\tilde{\beta}$ is unbiased, consistent, BLUE, efficient, & asymptotically efficient.

$\tilde{\sigma}^2$ is unbiased, consistent, efficient, & asymptotically efficient.

and, $\frac{(T-K)\tilde{\sigma}^2}{\sigma^2} \sim \chi^2_{T-K}$

Furthermore, $\tilde{\sigma}^2$ and $\tilde{\beta}$ are independent.

∴ Use appropriate t & F statistics to test hypotheses.

$$\text{e.g.} \quad \frac{\tilde{\beta}_i - \beta_i}{\sqrt{\tilde{\sigma}^2 (X'\Omega^{-1}X)^{-1}_{ii}}} \sim t_{T-K}$$

<u>Note</u>: Since $\tilde{\beta}$ is MLE (see p. 7) [minimizes SSE_{GLS}] we have its <u>asymptotic distribution</u> as well;

$$\sqrt{T}(\tilde{\beta} - \beta) \longrightarrow N\left[0, \sigma^2 \lim_{T\to\infty}\left(\frac{X'\Omega^{-1}X}{T}\right)^{-1}\right]$$

* No big deal since we know small sample distribution!

** It <u>is a big deal</u>, because when we <u>estimate $\hat{\Omega}$</u>, Again, the problem is to find a transformation, V, $\ni$ $V'V = \Omega^{-1}$. Then GLS is nice.

What if Ω is unknown? (the usual case)

1. assume it = something. (tacky)
2. estimate it.

Problem: Ω is $T \times T$; $\exists$ $T + \frac{1}{2}T(T-1)$ elements.
(it is symmetric)
$\rightarrow$ too many to estimate in general.

Solution(s):
- do OLS on $Y = X\beta + \epsilon$; get $\hat{\beta}$. [$\hat{\beta}$ unbiased & consistent; but inefficient.]

  let $\hat{\Omega} = e e'$.

In other words, look at the OLS residuals. Do they behave nicely?
- constant variance ?
- no autocorrelation ?

The residuals are a good Diagnostic to reveal any such problems.

$\therefore$ Look for problem [heterosked.?, autocorrel?], & then look for an appropriate V.

<u>More Rigorous Solution:</u> (than letting $\hat{\Omega} = ee'$)

It is "better" to assume $\Omega = f(\Theta)$
where Θ is $p \times 1$; i.e. Ω is determined by a few paramet

e.g. Heteroskedasticity ;
assume variance of ϵ_t is a $f(x_t)$, ...
e.g. $Var(Y) = f(\text{age})$

Autocorrelation ;
assume correlation between ϵ_t & ϵ_{t-1}
is $f(\text{something})$, ...

--- <u>More on this later!</u>

Then estimate Θ with $\hat{\theta}$.

<u>Thm:</u> If $\hat{\Theta}$ is a consistent estimate of Θ,
then $\hat{\Omega} = f(\hat{\Theta})$ is consistent estimate of Ω,
(by Slutsky Thm)

and then $\tilde{\beta}^* = (X'\hat{\Omega}^{-1}X)^{-1}X'\hat{\Omega}^{-1}Y$
is a <u>consistent</u> estimate of β,

and $\tilde{\beta}^*$ & $\tilde{\beta}$ will have the same
asymptotic distribution.

$\tilde{\beta}$ (it's the MLE)

$\Longrightarrow \therefore \tilde{\beta}^*$ will be <u>asymptotically efficient.</u>
[See Maddala, <u>ECON</u>, 1971.]

$\therefore$ better than $\hat{\beta}_{OLS}$