Matrix Algebra Review

Notation:

$\exists$	=	"there exists"
$\ni$	=	such that
$\forall$	=	for any
$\therefore$	=	therefore
lc	=	linear combination
lhs	=	left hand side
rhs	=	right hand side
iff	=	if and only if
$\sim$	=	is distributed
wrt	=	with respect to
rs	=	random sample

Things to emphasize:

MLE, sufficiency, consistency, efficiency, bias

small versus large sample properties,

Variance of lc of random variables $\Rightarrow$ $MSE(\tilde{\theta})$

Law of Large Numbers, Central Limit Theorem

how many, Linear Algebra? yes? – Review.
no? – wk hard & master!

I) Review of Matrix Algebra

1) Definitions

a) A matrix, $A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \equiv [a_{ij}]$ $\begin{matrix} i = 1,2,\ldots m \\ j = 1,2,\ldots n \end{matrix}$

$m \times n$

↑ rows ↑ columns

$\Rightarrow mn = $ # elements

b) A vector

If $m = 1$, a row vector, $b = [b_1, b_2, \ldots b_n]_{1 \times n}$

If $n = 1$, a column vector, $c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{m \times 1}$.

c) Scalar – a constant.

$(m = n = 1)$

In general, vectors follow the rules of matrix operations, but scalars do not.

d) Square Matrices $\Rightarrow$ $m = n \; (>1)$.

i) Symmetric Matrices

$$a_{ij} = a_{ji} \quad \forall\, i \neq j \; ; \; i,j = 1,2,\dots,n.$$

ii) Diagonal Matrices

$$a_{ij} = 0 \quad \forall\, i \neq j \qquad \begin{bmatrix} a_{11} & & \\ & a_{22} & 0 \\ 0 & & \ddots \\ & & & a_{nn} \end{bmatrix}$$

Note - A diagonal matrix is both "upper-triangular" & "lower-triang."

iii) Scalar Matrices

$$a_{ij} = 0 \quad \forall\, i \neq j \qquad \begin{bmatrix} \lambda & & \\ & \lambda & 0 \\ 0 & & \ddots \\ & & & \lambda \end{bmatrix}$$
$$a_{ij} = \lambda \quad \forall\, i = j$$

iv) Identity Matrices

$$a_{ij} = 0 \quad \forall\, i \neq j \qquad I_n = \underbrace{\begin{bmatrix} 1 & & \\ & 1 & 0 \\ 0 & & \ddots \\ & & & 1 \end{bmatrix}}_{n \times n}$$
$$a_{ij} = 1 \quad \forall\, i = j$$

Note: Each class of matrices is a subset of the previous class. Thus, any proof or property that holds for a more general class will hold for the more restrictive classes.

TRACE

ex. 1 Square matrices have traces,

$$tr(A) = \sum_{i=1}^{n} a_{ii} \quad \forall (i=j)$$
$$= \text{the } \Sigma \text{ of the principle diagonal}$$

$\rightarrow$ applies to i) – iv)

2) Operations & Properties

a) Equality, $A_{m\times n} \equiv B_{m\times n}$ (same dim.)

if $a_{ij} = b_{ij} \quad \forall ij$

b) Addition, $C_{m\times n} \equiv A_{m\times n} + B_{m\times n}$

if $c_{ij} = a_{ij} + b_{ij} \quad \forall ij$

i) Commutative Law holds $\Rightarrow A+B = B+A$

ii) Associative Law holds $\Rightarrow (A+B)+C = A+(B+C)$

~~iii) Distributive Law holds~~

c) Multiplication by a scalar, λ.

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \ddots & & \vdots \\ \vdots & & & \\ \lambda a_{m1} & \cdots & & \lambda a_{mn} \end{bmatrix} = \lambda [a_{ij}] \quad \forall ij$$

A can be any dimension.

d) Matrix Multiplication

$$C_{m\times p} \equiv A_{m\times n} B_{n\times p} \qquad (\text{req. \# col. } A = \text{\# rows } B)$$

$$\text{if} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\text{e.g.} \Rightarrow c_{11} = \sum_{1}^{n} \underset{\substack{\uparrow \\ 1^{st}\text{ row} \\ \text{of } A}}{a_{1k}} \underset{\substack{\uparrow \\ 1^{st}\text{ column} \\ \text{of } B}}{b_{k1}}$$

$$= a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1}$$

$$\Downarrow$$

$$AB = \underset{m\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ & & & \end{bmatrix}} \underset{n\times p}{\begin{bmatrix} b_{11} & \cdots \\ b_{21} & \cdots \\ \vdots & \\ b_{n1} & \cdots \end{bmatrix}} = \underset{m\times p}{\begin{bmatrix} c_{11} & \cdots \\ \vdots & \\ & \end{bmatrix}} = C$$

ex.

$$\underset{2\times 3}{\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ -1 & 3 \end{bmatrix}}$$

*** i) Commutative Law does <u>not</u> hold $\Rightarrow AB \neq BA$

- Premultiplication $\neq$ Postmultiplication
- ~~There are some~~ exceptions

ii) Associative Law holds $\Rightarrow (AB)C = A(BC)$

iii) Distributive Law holds $\Rightarrow A(B+C) = AB + AC$

Comment: The method used in Johnston, pp. 72-73 of proving these last two laws of matrix multiplication is useful.

Notes: ① λI is a scalar matrix

~~[illegible] diagonal matrix~~

e) Transposition

If $A = [a_{ij}]$, then $A' = [a_{ji}]$.

— [flip about the diagonal]

Properties:

i) If A exists, A' exists

ii) $(A')' = A$

iii) $(A+B)' = A' + B'$

iv) $(AB)' = B'A'$ $\quad (AB)^T = B^T A^T$

~~[illegible]~~

Definitions:

Symmetry — If $A' = A$

Idempotency — If $AA = A$ (& not also $A' = A$)

Orthogonality — If $AB = 0$, A is orthog. to B.

Dot product

f) Determinants

Definition - A scalar quantity associated w/ square matrices, commonly noted $|A|$

$$|A| \equiv \begin{cases} a_{i1}c_{i1} + a_{i2}c_{i2} + \dots + a_{in}c_{in} & (i^{th} \text{ row}) \\ a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} & (j^{th} \text{ col.}) \end{cases}$$

exp. by:

where: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

and the co-factors, $c_{ij} \equiv (-1)^{i+j} \, \underbrace{|A_{ij}|}_{\text{the Det. of the } ij^{th} \text{ minor } (*)}$

Note - $(-1)^{i+j} = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$

Comment ① - It is interesting that the value of the determinant is the same, regardless of which row or column is chosen to expand along.

Comment ② - Det of a 2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

* Example -

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

(1st col.) $|A| = 3\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$

$= 3(0) + 2(1) + (-1)$

$= \underline{1}$

(1st row) $|A| = 3\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 0 + \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}$

$= \underline{1}$

w/ row replacement:

Multiply $\underline{3^{rd}}$ col. by (-3) & add to 1st & then replace 1st:

$$\begin{bmatrix} 0 & 0 & 1 \\ -5 & 1 & 1 \\ 4 & -1 & -1 \end{bmatrix} \Rightarrow |A| = 5 - 4 = \underline{1}$$

$$\begin{matrix} 3 & 0 & -3 \\ -2 & 1 & -3 \\ 1 & -1 & 3 \end{matrix} \longrightarrow \begin{matrix} 0 & 0 & \cancel{-3} & 1 \\ -5 & 1 & \cancel{-3} & 1 \\ 4 & -1 & \cancel{3} & -1 \end{matrix}$$

$$\frac{-9}{-3}$$

Properties of Determinants:

① Interchanging any two rows (or columns)
$\Rightarrow$ New det. $= -|A|$.

② If any two rows (~~or~~ columns) are identical, then $|A| = 0$

* Definition - A matrix is said to be singular if its determinant is zero; In this case, either its rows and/or columns are not linearly independent of one another.

③ If any row (or column) is multiplied by a scalar, λ,
$\Rightarrow$ New det $= \lambda |A|$

Corollary: $|\lambda A| = \lambda^n |A|$ $\qquad A - (n \times n)$

④ If any row (or column) is replaced by a linear combination of rows (or columns), there is no change in $|A|$. (Note - the row or column being replaced must be included in the linear combination.)

Corollary ② & ④: If any row (or column) is a linear combination of other rows (or columns), $|A| = 0$; that is, A is singular.

Comment: The technique of row (or col.) replacement may be used to simplify the computation of a determinant. (See ③ of previos example.)

→ get in form with all but 1, 0's

⑤ $|A'| = |A|$

⑥ If A & B are square matrices of the same dimension, then $|AB| = |A||B|$.

Corollary: If AB is nonsingular

⇒ both A & B are nonsingular.

g) Inverses

Definition - A square matrix A, is said to have an <u>inverse</u>, A^{-1},

$$\text{if} \quad A^{-1}A = AA^{-1} = I.$$

Computation -

$$A^{-1} = \frac{1}{|A|}(\text{adjoint } A).$$

⇒ A must be nonsingular for A^{-1} to exist

$$\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

matrix of transposed co-factors of A

Example -

$$\text{if } A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$c_{11} = (-1)^2 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$c_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$\vdots$$

<u>Check</u>: $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓.

Properties – [A, B $_{n\times n}$ and nonsingular]

① $(AB)^{-1} = B^{-1}A^{-1}$ – ~~requires both A & B square, and AB nonsingular ⇔ A & B nonsingular~~

② $(A')^{-1} = (A^{-1})'$

③ $|A^{-1}| = \dfrac{1}{|A|}$

Application – "Cramer's Rule"

Given a system of n equations:

$$\underset{n\times n}{A}\ \underset{n\times 1}{x} = \underset{n\times 1}{h}$$ w\ x's $(n\times 1)$ unknown – [want to solve for x's]

$$\Rightarrow x = A^{-1}h$$ w\ A nonsingular

$$= \frac{1}{|A|}\cdot\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn}\end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n\end{bmatrix}$$

$$\Rightarrow x_1 = h_1c_{11} + h_2c_{21} + \ldots h_nc_{n1} \,/\, |A|$$

$$= \frac{|h \vdots A_{2-n}|}{|A|}$$ 1st col. expansion w\ h's instead of a's!

(explain with examples)

$$= \begin{vmatrix} h_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ h_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & & \\ h_n & a_{n2} & \cdots & \cdots & a_{nn}\end{vmatrix} \Big/ |A|$$ ← usual form

Or, in general,

$$x_i = \left| A_{1-(i-1)} \,\vdots\, h \,\vdots\, A_{(i+1)-n} \right| \Big/ |A|$$

Example -

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 \qquad\quad - x_3 &= 0 \\ -2x_1 - 2x_2 - x_3 &= -4 \end{aligned}$$

(3 equations, 3 unknowns)

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -2 & -2 & -1 \end{bmatrix}, \quad x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad h = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

<u>By Cramer's Rule:</u>

$$x_1 = \frac{\begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & -1 \\ -4 & -2 & -1 \end{bmatrix}}{|A|} = \frac{-(-1)\begin{vmatrix} 2 & 2 \\ -4 & -2 \end{vmatrix}}{\underbrace{-2\begin{vmatrix} 2 & -1 \\ -2 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}}_{\text{exp. about } 2^{nd} \text{ column}}} = \frac{4}{+8-6} = +2$$

$$x_2 = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -2 & -4 & -1 \end{vmatrix} = \frac{-2(-2-2)+4(-1-2)}{2} = \frac{+8-12}{2} = -2$$

$$x_3 = \frac{1}{2}\begin{vmatrix} 1 & 2 & 2 \\ 2 & 0 & 0 \\ -2 & -2 & -4 \end{vmatrix} = -\frac{1}{2}\cdot 2(-8+4) = 4$$

$$\Rightarrow \quad x = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} = \checkmark$$

Alternative Method - "Gaussian Elimination"

① Create an "augmented" matrix: [A : b]

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & -1 & 0 \\ -2 & -2 & -1 & -4 \end{bmatrix}$$

② Simplify into "echelon form"* w/ row & column operations:

$\boxed{\text{1's down diag. 0's off diag.}}$ *

$\boxed{\text{The echelon matrix is the square matrix w/o the 4th column.}}$

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} & \begin{array}{l} ① \\ ② \\ ①+③ \end{array} \end{array}$$

$$\begin{array}{ccl} (-1) * & \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} & \begin{array}{l} \text{new } ③ \\ ① + \text{new } ③ \\ 2\,\text{new } ③ + ② \end{array} \end{array}$$

×(-1) for switching

$$\begin{array}{ccl} (-1)(-1) * & \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{l} ① \\ \text{[redacted]}\ \tfrac{1}{2}[②+③] \\ -③ \end{array} \end{array}$$

Comment: The general technique is to transform the l.h. partition into an identity matrix (∴, the solution is obvious). (in 4th column)

*: ~~echelon form = an "upper triangular matrix," w/ 1's on the principal diagonal~~

Intuition:

$$\begin{array}{ll} \text{Start} - & [A : h] \\ & \ \downarrow \quad \downarrow \\ \text{Finish} - & [I : x] \end{array}$$

$$\boxed{\begin{aligned} Ax &= h \\ A^{-1}Ax &= A^{-1}h \\ Ix &= x \end{aligned}}$$

Intuition: As A is transformed into I, h is transformed into x!

h) Rank

Definition – the maximum # of linearly independent rows & columns of a matrix.

Theorems:

① The rank of a matrix does not change w/ row & col. operations.

② The rank of a matrix in "echelon form" is its number of nonzero row vectors.

Corollary: Max. rank = min(m,n).

③ A square matrix, $A_{n \times n}$ is nonsingular iff* it is of Rank n (i.e., full row rank).

~~④ R(echelon form) ≤ R(augmented matrix).~~

* 2> => the converse is true

Computation –

ex. 1) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

$[-2①+②] \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow$ Rank = 2

Note – $[\ \ ]_{2\times 3}$, max. is two, since any of the columns **must** be a lin. comb. of the other two –

e.g. $1^{st} = \frac{3}{2}\cdot 2^{nd} + \frac{1}{2}\cdot 3^{rd}$ ✓

⋮

ex. 2) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ $1(-8)-2(-4)+1(0)$

$-8+8$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ ①−②
 2①−③

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Rank 2

$Ax = b$

$x = A^{-1}b \rightarrow$ If A is nonsingular, $\det(A) \neq 0$

Rank$(A) = n$ then $\forall$.

Note on the Existence & Uniqueness of Solutions – (in systems of n linear equations)

~~① If R(augmented) > n ⇒ ∄ solution~~

~~Overidentified~~

~~ex. $x_1 = 1$ and $x_1 = 2$~~ $\cancel{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$ $\cancel{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow R(2) > n}$

① If R(echelon) = R(augmented) $= n$,

$\exists$ a unique solution.

ex. $x_1 + x_2 = 0$

$x_1 + 2x_2 = 1$

Augmented Matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad ②-①$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad ①-②$$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Exactly Identified ☆

$$* \begin{cases} R(e) = 2 \\ R(a) = 2 \end{cases}$$

(2) If $R(\text{echelon}) = R(\text{augmented}) < n$

$\Rightarrow$ solution $\exists$, but it will <u>not</u> be unique.

ex.

$$x_1 + x_2 = 2$$
$$2x_1 + 2x_2 = 4$$

∞ Solutions since $\parallel$ lines

Underidentified

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow R(\text{aug.}) = R(\text{ech.}) = 1 < n$$

[not enough info. in the 2 equ's to solve for unique x_i]

Solution: $x_1 = 2 - x_2$

(3) If $R(\text{echelon}) < R(\text{augmented})$

$\Rightarrow$ solution $\nexists$.

ex.

$$x_1 + x_2 = 2$$
$$2x_1 + 2x_2 = 3$$

[almost same as (2)]

Overidentified/Inconsistent

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} R(\text{aug}) = 2 \\ R(\text{ech}) = 1 \end{matrix}$$

<u>Comment</u>: This concept is particularly important when dealing with simultaneous equations models, or models w/ restrictions.

Importa...

☆ i) Quadratic Forms & Positive (Semi-) Definite Matr

[$x_{n\times 1}$ - column vector
$A_{n\times n}$ - matrix

Properties

① If A is symmetric, $x'Ax$ is a quadratic form
☆ see opposite ☆

② If $x'Ax > 0$ $\forall x \neq 0$ $\Rightarrow$ A is p.d.
* see opposite (bottom)

$X'AX>0 \Rightarrow AX \neq 0 \Rightarrow \text{Rank}(A)=n$

For $X'A^{-1}X$, $\forall X \neq 0$
let $AY=X$
$X'A^{-1}X = Y'A'A^{-1}AY = Y'A'Y$
A is p.d $\Leftrightarrow$ A' is p.d
A^{-1} is p.d

③ If A is p.d., then A is nonsingular.

④ If A is p.d., then A^{-1} is p.d.

⑤ If $x'Ax \geq 0$ $\forall x \neq 0$ $\Rightarrow$ A is p.s.d.

⑥ If A is p.s.d., then A is p.d.
iff A is nonsingular.

?

e.g. matrix w/ ... could be ... but not ...

(very useful) → ⑦ A is p.s.d. iff $\exists$ a matrix $C \ni A = C'$
* again; diagonal elements ≥ 0 -- they are squar
emphasize con

⑧ If A is p.s.d., $C'AC$ is p.s.d.
Note: C need not be square, as long as dimensions

⑨ If A & B nonsingular,
$A-B$ is psd iff $B^{-1}-A^{-1}$ is psd.

[e.g. $3 >$ ...
$\Leftrightarrow$ ...

⑩ Any covariance matrix is psd.
(most reasonable cov matrices are p.d.)

(formal defn):

p.d. $\Rightarrow$ all principal minors are positive.

p.s.d. $\Rightarrow$ " " " " nonnegative.

Principal Minors: $|a_{11}|$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, ...

~~determinants formed by deleting the ith row and ith column, where i is any i = 1, ..., n~~

~~expansion along the diagonal~~

~~$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$~~

~~"1st order" $a_{11}, a_{22}, \ldots a_{nn}$~~

~~"2nd order" $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ or in fact $\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$ $\forall\, i \neq j$~~

~~a rather strong, though not rare property~~

Comment:

This is useful when comparing the difference between matrices –

ex. If $Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) =$ a p.s.d. matrix

then estimator $\hat{\beta}_2$ will be efficient (i.e. less variance) rel. to est. $\hat{\beta}_1$.

j) Matrix Differentiation Rules

1) Scalar w.r.t. a vector –

λ, scalar

$$\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_p \end{bmatrix}, \text{ a } (p \times 1) \text{ vector}$$

$$\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$\leftarrow$ * Define: $$\frac{\partial \lambda}{\partial q} \equiv \begin{bmatrix} \partial\lambda/\partial q_1 \\ \partial\lambda/\partial q_2 \\ \vdots \end{bmatrix}$$ (also a $(p \times 1)$ vector)

and $$\frac{\partial^2 \lambda}{\partial q \partial q'} \equiv \left[\frac{\partial\left(\frac{\partial \lambda}{\partial q}\right)}{\partial q'} \right]$$ $\leftarrow$ [deriv.'s of $p \times 1$ vector wrt $1 \times p$ vector $\rightarrow (p \times p)$]

$$= \begin{bmatrix} \partial^2\lambda/\partial q_1 q_1 & \partial^2\lambda/\partial q_1 q_2 & \cdots \\ \partial^2\lambda/\partial q_2 q_1 & \partial^2\lambda/\partial q_2 q_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

2) Vectors – α & q $(p \times 1)$

Define: $$\frac{\partial(\alpha' q)_{1\times 1}}{\partial q} = \frac{\partial(q'\alpha)_{1\times 1}}{\partial q} = \alpha$$, again a $(p \times 1)$ vector

$\boxed{\alpha' q = q'\alpha \text{ a scalar!}}$

3) Symmetric Matrices $(B)_{p \times p}$ w.r.t. a vector–

Define ≡ $\frac{\partial(q' B q)_{1\times1}}{\partial q} = 2Bq$ again, (a $p\times1$ vector)

"Proof":

↳ ex. Let $p = 2$

$$(q' B q) = [q_1, q_2]\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= b_{11}q_1^2 + 2b_{12}q_1q_2 + b_{22}q_2^2 \quad \text{(a scalar)}$$

$(b_{12} = b_{21})$*

$$\Rightarrow \frac{\partial}{\partial q_1} = 2b_{11}q_1 + 2b_{12}q_2$$

$$\frac{\partial}{\partial q_2} = 2b_{12}q_1 + 2b_{22}q_2$$

$$\Rightarrow \frac{\partial}{\partial q} = \begin{bmatrix} \frac{\partial}{\partial q_1} \\ \frac{\partial}{\partial q_2} \end{bmatrix} = 2\begin{bmatrix} b_{11}q_1 + b_{12}q_2 \\ b_{12}q_1 + b_{22}q_2 \end{bmatrix} = 2Bq.$$

(induction → $n \times n$ case) → Q.E.D.

[intuitive]

Exercises

1. Prove or Disprove the following.

(½ pt.) If A and B are $n \times n$ matrices, then $(A+B)(A-B) = (A-B)(A+B)$

2. Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$

(½ pt.) a) find Ab, AC, $(Ab)'$, and $(AC)'$

(½ pt.) b) List A', b', and C'

(½ pt.) c) find $b'A'$ and $C'A'$

3. Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 8 & -9 \\ 1 & 4 & 3 \end{bmatrix}$

(1 pt.) Given $AB = C$, find the values a through f in matrix B.

4. Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$

(½ pt.) a) find AD and $(AD)^{-1}$

(½ pt.) b) find A^{-1} and D^{-1}

(½ pt.) c) find $(D^{-1}A^{-1})$

(½ pt.) d) find $(A')^{-1}$

5. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(½ pt.) a) find AJ and JA
(½ pt.) b) find AK and KA
(½ pt.) c) find AL and LA
(½ pt.) d) find MA and AM and $M'A$

6. Let $a = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\lambda = x_1 - 4x_2^3 + 2x_3^2$

and $B = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(½ pt.) a) find $\frac{d\lambda}{dx}$
(½ pt.) b) find $\frac{d(a'x)}{dx}$
(½ pt.) c) find $\frac{d(x'Bx)}{dx}$

7. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 9 \\ 7 & 6 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(1 pt.)

Find $|A|$, $|E|$, and $|AE|$

8. Suppose that

$$\underset{T\times 1}{Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}}, \quad \underset{T\times K}{X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & & & \\ x_{T1} & x_{T2} & \cdots & x_{TK} \end{bmatrix}}, \text{ and } \underset{K\times 1}{\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{bmatrix}}$$

(1 pts.) a) Show that $\dfrac{d(Y'X\beta)}{d\beta} = \dfrac{d\,\beta'X'Y}{d\beta} = X'Y$

(1 pts.) b) Show that $\dfrac{d(\beta'X'X\beta)}{d\beta} = 2X'X\beta$

9. Let $M = [I_T - X(X'X)^{-1}X']$, where X is $T\times K$.

(½ pt) a) What are the dimensions of M?
(½ pt) b) Show that M is symmetric.
(½ pt) c) Show that M is idempotent.
(½ pt) d) Show that M is orthogonal to X.

10. Let A be $m\times n$ and B be $n\times m$.

(1 pt.) a) Prove that $\text{Trace}(AB) = \text{Trace}(BA)$
(½ pt.) b) Show that $\text{Trace}(M) = T-K$ where M is as in question 9.

TOTAL = **15** pts.