EG, Ch. 22; Options

I. Overview.

A. Definitions.

1. **Option** - contract in entitling holder to *buy/sell* a certain asset *at or before* a *certain time* at a *specified price*.

   Gives holder the right, but not the obligation, to do something.
   Call - ... *buy* ...
   Put - ... *sell* ...

   European Option - ... *at* a certain time (not before) ...
   American Option - ... *at or before* a certain time ...

   Expiration/Maturity - the *certain time* (T).
   Exercise Price/Strike Price - the *specified price* (E).

   A Call is in-the-money (itm) if $S > E$;
   A Call is at-the-money (atm) if $S = E$;
   A Call is out-of-the-money (otm) if $S < E$;

   Warrant - option issued by parent corporation or fin. institution rather than by another investor, to achieve some objective.

B. Combinations.

1. Synthetic Call (Put-Call Parity).
2. Writing a Covered Call.
3. Straddle, Strangle.
4. Spreads (Bull, Bear, Butterfly).

C. Uses.

1. Options can be combined to create any payoff pattern desired; Possible variations are only limited by imagination.

2. Options have substantial "inherent leverage."
   These characteristics make options powerful and useful tools for speculation or hedging.

II. Mechanics of Options Markets.

A. Option Payoffs, Strategy, & Intuition.

1. If you think S will ↑, buy a call;

   a. If right (S↑), S > E, exercise;
   can buy @ E, sell @ S, worth (S - E).

   b. If wrong (S↓), S < E, lose price of call.

2. If you think S will ↓, sell a call;

   a. If right (S↓), S < E, keep price of call.

   b. If wrong (S↑), S > E, will be exercised;
   must sell @ E, buy @ S, lose (S - E).

3. If you think S will ↓, buy a put;

   a. If right (S↓), S < E, exercise;
   can sell @ E, buy @ S, worth (E - S).

   b. If wrong (S↑), S > E, lose price of put.

4. If you think S will ↑, sell a put;

   a. If right (S↑), S > E, keep price of put.

   b. If wrong (S↓), S < E, will be exercised;
   must buy @ E, sell @ S, lose (E - S).

5. Summary:

	Stocks	Calls	Puts	
Think S will ↑?	Buy	Buy	Sell	
Think S will ↓?	Sell	Sell	Buy	
Buy if you think:	S↑	S↑	S↓	Unlimited upside
Sell if you think:	S↓	S↓	S↑	Unlimited downside

6. Offsetting Orders.

   a. Anyone who is *long* an option can close out position by issuing an offsetting order to *sell* the same option.

   b. Anyone who is *short* an option can close out position by issuing an offsetting order to *buy* the same option.

   c. What happens to open interest when a contract is traded?

      i. If neither party is offsetting an existing position, open interest increases by one.

      ii. If one party is offsetting an existing position, open interest stays the same.

      iii. If both parties are offsetting existing positions, open interest goes down by one.

B. Warrants, Intrinsic & Extrinsic Value, and OTC markets.

1. Option trade is like a side bet; has no effect on value of the firm (S).

   a. Warrant is different – issued by parent corporation or fin. inst.

      a. When issued, firm receives cash flow (price of warrant).

      ii. When exercised, affects cash flow & ownership interest:
         - new shares issued
         - firm receives cash flow (E);

      iii. Warrant issuance & exercise affects value of firm (S)!
      More difficult to value warrants.

2. Call option is:

	itm	atm	otm
if:	S > E	S = E	S < E
if:	(S-E)>0	(S-E)=0	(S-E)<0

   a. **Intrinsic value** of call = Max{S-E, 0}.
   (payoff if exercised now)

   b. **Intrinsic value** of put = Max{E-S, 0}.

   c. If transactions costs = 0,
   An itm option will always be exercised @ expiration (if not early).

      i. It may be optimal for holder to wait or sell,
      rather than exercise early.

      ii. In this case, option also has **Extrinsic value** (or 'time value').

   a. **Total value** of option = (**Intrinsic value**) + (**Extrinsic value**).

3. OTC Markets.

   a. Not all options trade on exchange.

   b. FC options also traded on OTC Interbank mkt between fin. inst.'s,
   Or between fin. inst. & its corporate clients.

   c. Interest rate options traded in many forms OTC through banks.
   e.g., caps, floors, collars, swaps, …

C. Stock Splits, Stock Dividends, & Cash Dividends.

1. Option terms are adjusted when there is a *stock split* or *stock dividend*.

2. Valuation of European call is simple (Black/Scholes).
   What about American call on stock that changes value?

   a. Suppose company makes 2 for 1 **stock split**.
      Should make S↓ by half; affect value of option.

   b. Suppose company declares a **stock dividend**.
      Should make S↓ by certain amount.

   c. m for n **stock split** should make S↓ by (n/m).
      e.g. 3 for 1 split; (new S) = (1/3)(old S).

   d. **Stock dividends** are similar.
      e.g. 20% stock dividend like a 6 for 5 stock split;
      (new S) = (5/6)(old S)

3. Most exchange-traded options are protected by automatic adjustments in the number of shares to be exchanged (N) and the exercise price (E).

   a. E is adjusted by (n/m), to match expected change in S.
      N is adjusted by (m/n), to match dilution in value of S.

   b. e.g. 3 for 1 split; new option is for 300 shares @ E' = (1/3)(old E).
      e.g. 20% stk.div; new option is for 120 shares @ E' = (5/6)(old E).

4. Cash Dividends.

   a. Suppose company declares a $1 dividend.
      What happens to S on ex-date?

   b. Early OTC options were protected against dividends.
      (E was reduced by amount of dividend.)

   c. Exchange traded options are NOT dividend protected.
      This means an American call loses value on ex-date.

III. Complex Trading Strategies Involving Options -- Basic Combinations.

A. Call and Put Options can be combined with other building blocks (Stocks and Bonds) to provide almost any payoff pattern desired.

1. For simplicity, assume European call & put options with the same expiration date & underlying asset.
2. Already know payoff patterns for buying & selling calls & puts.
3. Consider payoffs for long & short positions on:
   a. Stocks.
   b. Bonds.

B. Buy Stock (+S) and Buy a Put (+P): (S + P) -- Protective Put.

1. If S ↑, payoff ↑; If S ↓, payoff flat.
   a. Gains on put offset losses on stock).
   b. Reduces return at higher S in exchange for hedging against losses.

C. Buy Bond (+B) and Buy a Call (+C): (B + C).

1. If S ↑, payoff ↑; If S ↓, payoff flat.
   a. Bond payoff fixed; just shifts up Call payoff.
   b. Identical pattern to C; (S + P) = (B + C).

D. Short a Call (-C) and buy the Stock (+S) – Covered Call write.

1. If S ↑, payoff flat (gains on stock offset by losses on call)
2. If S ↓, payoff S ↓ (lose on stock, call OTM).
3. Like being short a put.

E. Spreads.

1. Bull, Bear, Butterfly, Calendar, Diagonal Spreads.

F. Other Combinations.

1. Straddle, Strip, Strap, Strangle.

IV. Basic Properties of Options.

A. Notation and Assumptions:

1. Notation:

S_0: today's stock price;

S_1: stock price at maturity of option;

E: exercise price of option;

T: time to maturity of option (fraction of year);

σ_S: volatility of stock returns;

R: riskfree rate of interest;

C: value of call option to buy one share;

P: value of put option to sell one share.

Note: R = nominal rate of interest (not real) > 0.

B. Assumptions.

1. No transactions costs.
2. All net profits are subject to same tax rate.
3. Can borrow or lend at same riskfree rate (R).
4. Arbitrageurs take advantage of any opportunities.

C. Economic Factors Affecting Option Prices.

$$\text{Value of a Call} = C = f\{\overset{+}{S}, \overset{-}{E}, \overset{+}{T}, \overset{+}{R}, \overset{+}{\sigma_S}, \overset{-}{D}\}.$$

Call option is more valuable if:

1. underlying stock price (S) increases.
2. you have the right to buy at a lower strike price (E).
3. there is a longer time to maturity (T).
4. the riskfree rate (R) increases.
5. the underlying stock price (S) is more volatile (σ_S).
6. any dividend paid during option life is smaller (D).

7. Summary of How Factors Affect Prices of Options.

Variable	European Call (C)	European Put (P)
S	+	-
E	-	+
T	+(?)	+
R	+	-
σ_S	+	+
D	-	+

(?)Long term call usually costs more than short term call.
But suppose dividend is paid after short term call expires.
Holder of short term European call will get the dividend.
Holder of the long term European call will not.
So short term call may be worth more than long term call.

D. **Lower Bound for European Call** on non-dividend paying stock.

Consider two portfolios.

Portfolio A: Buy one call and E bonds each paying $1 at T.
Cash flows today: -C -E/(1+R) (will need $E at exp.)

Portfolio B: Buy one share of stock.
Cash flows today: $-S_0$

Portfolio	Cash Flows Today	Value at Expiration . If $S_1>E$	If $S_1<E$
Portfolio A:	-C	S_1-E	0
	-E/(1+R)	E	E
Total:	-C - E/(1+R)	S_1	E
Portfolio B:	$-S_0$	S_1	S_1

Obs. #1: If S increases above E, both portfolios pay S_1;
If S decreases below E, Portfolio A does better (hedged).

Thus Portfolio A is worth at least as much as B.

$$C + E/(1+R) \geq S_0 \quad \text{or} \quad C \geq S_0 - E/(1+R)$$

European Call cannot sell for less than $[S_0 - E/(1+R)]$.

Obs. #2: If R higher, E/(1+R) lower; call is more valuable;
pay less today for bond that promises $E at expiration.
-- Don't have to tie up as much $ today, if R increases.

Obs. #3: American call will not be exercised early (if no dividend).

$$C \geq S_0 - E/(1+R) > S_0 - E.$$

Can exercise American call early, and receive $S_0 - E$;
Or, can sell American call, and receive C [> $S_0 - E$] !
-- American call acts like European call (if no dividend)!
-- American and European call worth the same.

E. **Put-Call Parity**.

For European options, there is a fixed relationship between the price of put and call options with the same maturity and written on the same underlying security.

Recall: $S + P = B + C$ or $S + P - B = C$.

Consider the combination, $S + P - B$;

Buy one share of stock $(+S_0)$, buy one put $(+P)$, and sell a bond $(-B)$ for $E/(1+R)$, maturing at expiration.

Portfolio	Cash Flows Today	Value at Expiration If $S_1>E$	Value at Expiration If $S_1<E$
Portfolio A (buy synthetic call):			
buy stock	$-S_0$	S_1	S_1
buy put	$-P$	0	$E-S_1$
sell bond	$+E/(1+R)$	$-E$	$-E$
Total:	$-S_0 - P + E/(1+R)$	S_1-E	0
Portfolio B (buy call):	$-C$	S_1-E	0

Outcomes are identical.
Thus, initial cost of Portfolio A must be same as call:

$$-C = -S_0 - P + E/(1+R)$$

$$C = S_0 + P - E/(1+R) \qquad \text{Put-Call Parity.}$$

F. **Equity as a call option**.

Equity in a levered firm is a call option on the Value of the firm (V).

Simplified Example.

1. Firm has two sources of capital – Debt & Equity.
2. Debt is zero coupon bond with face value = D;
   D will be paid in T years, or firm defaults.
3. Debt is secured by firm's assets.
4. Firm pays no dividends.
5. Ignore taxes and bankruptcy costs.
6. At maturity, V is divided among bondholders & shareholders.
7. For call option, underlying asset is a share of stock, S.
8. For this illustration, the share of stock is the call option,
   and the underlying asset is the value of the firm, V.
   a. If at maturity, $V > D$, this "option" is ITM;
      Shareholders will exercise their option by
      paying debtholders D, and keeping the rest (V - D).
   b. If at maturity, $V < D$, this "option" is OTM;
      Shareholders will default on debt & give firm to debtholders.
   c. At maturity, shareholder's wealth is:
      $W = \max\{0, V - D\}$. A call option on V with strike, D!

V. Option Valuation.

A. Overview.

1. Economic characteristics of an option that give it value.

   \+ - + + +

   a. Value of a Call = C = $f(S, E, T, R, \sigma_S)$.

      Call Option is more valuable if:

      i. the underlying stock price (S) increases;
      ii. you have the right to buy at a lower strike price (E decreases);
      iii. the time to maturity (T) increases;
      iv. the riskfree interest rate (R) increases;
      v. the underlying stock price is more volatile (σ_S increases).

2. Binomial Option Pricing Model.

   a. 1-Period Model: $C = [C_u p + C_d (1-p)]/(1+R)$,

      where $p = [(1+R)-d]/(u-d)$, and R = riskfree rate per annum;

      S_0 = today's stock price, follows a Binomial distribution;
      S_0 may increase by the ratio, u, with 'probability' p,
      or decrease by the ratio, d, with 'probability' (1-p);

      C_u = value of call if S_0 increases;

      C_d = value of call if S_0 decreases.

   b. 2-Period Model: $C = [C_{uu} p^2 + 2C_{ud} p(1-p) + C_{dd} (1-p)^2]/(1+R/2)^2$.

      where C_{uu} = value of call if S_0 increases twice (by u each time);

      C_{dd} = value of call if S_0 decreases twice (by d each time);

      C_{ud} = value of call if S_0 increases by u & then decreases by d.

   c. N-Period Model: $C = S_0 B[a,N,p'] - E/(1+R/N)^N * B[a,N,p]$.

      where a = number of increases out of N trials
      required for the call to finish in-the-money ($S > E$);

      B[a,N,p] = cumulative probability of getting ≥ a increases
      out of N trials (probability that call will finish itm);

      $p' = [u / (1+R/N)] p$.

3. Black/Scholes Option Pricing Model.

   a. Limiting case of N-Period Binomial Model.

      $$C = S_0 N(d_1) - E e^{-RT} * N(d_2);$$

      where $d_1 = [\ln(S_0/E) + (R + \frac{1}{2}\sigma_S^2)T]/\sigma_S(T)^{\frac{1}{2}}$; and $d_2 = d_1 - \sigma_S(T)^{\frac{1}{2}}$.

B. **One-Period Binomial Model**.

1. Introduction -- An Example. Assume:

   a. S_0 = \$50, E = \$50, call expires in *one year*;

   b. S will either increase or decrease by 50%
   (i.e. S_1 = either uS_0 = \$75 or dS_0 = \$25,
   where u = 1.5 and d = .5);

   c. R = .25

2. Constructing a riskless "hedge portfolio":

   a. sell two calls;

   b. buy one share of stock.

Hedge Portfolio	cash flow today	outcome at end of pd S_1 = \$25	S_1 = \$75
Sell 2 calls	+2C	0	-50
Buy 1 share	-50	+25	+75
Total:	+2C-50	+25	+25

3. Solving for the value of the call.

   a. Note: (investment) = -(flows today) = -2C+50.

      i. At end of period, the payoff is \$25 regardless.
      This portfolio should yield the riskless rate (R).

      → (investment) * (1+R) = outcome

      → [-2C+50] * (1.25) = \$25
      → -2C+50 = \$25/1.25 [= \$20]
      → -2C = -\$30
      → C = \$15

   b. What if C = \$20? Then [-2C+50] * 1.25 → \$25
   [-40+50] * 1.25 → \$25
   [\$10] * 1.25 → \$25
   (earn > 1+R with no risk).

4. The hedge portfolio: buy Δ shares of stock for each call written.

   a. Define:
   Δ = hedge ratio; the number of shares of stock per call written that makes the payoff riskfree;
   C = value of call today;
   C_u = value of call if S_1 increases to uS_0 = max{uS_0-E, 0};
   C_d = value of call if S_1 decreases to dS_0 = max{dS_0-E, 0}.

   b. In this example, $\Delta = 1/2$; C_u = max{75-50,0} = \$25; C_d = \$0.

Portfolio	cash flow today	outcome at end of pd. $S_T = 25$	$S_T = 75$
Sell one call:	+C	$-C_d$	$-C_u$
Buy Δ shares:	$-\Delta S$	$+\Delta dS_0$	$+\Delta uS_0$
Total:	$C-\Delta S$	$-C_d + \Delta dS_0$	$-C_u + \Delta uS_0$

   c. To be a riskless "hedge portfolio," cash flows at end of period must be same, whether S ↑ or ↓:

   $-C_d + \Delta dS_0 = -C_u + \Delta uS_0$.

   d. Solving for Δ, the number of shares per call in hedge pf;

   $\Delta = (C_u - C_d) / (uS_0 - dS_0)]$ = (\$25 - \$0) / (\$75 - \$25) = ½ .

   e. Note: Hedge ratio = dC/dS = (Δ option price)/(Δ stock price) as we move between the nodes.

   f. Define **Delta**: The ratio of the change in the price of an option to the change in the price of the underlying stock.

   This is the hedge ratio, Δ; The number of shares we should hold for each call written to create a riskless hedge portfolio. Construction of this hedge is called delta hedging (more later).

5. General formula for One-Period Binomial Model

a. Fact: Hedge portfolio should yield the riskless rate;
(investment) * (1+R) = riskless outcome.

b. If we buy Δ shares of stock and sell one call, we have:

(initial investment) = (ΔS-C).

c. Thus, $(\Delta S_0 - C)(1+R) = -C_d + \Delta d S_0$ [or $= -C_u + \Delta u S_0$]

Thus: $(C - \Delta S_0)(1+R) = C_d - \Delta d S_0$

$C(1+R) = C_d - \Delta d S_0 + \Delta S_0(1+R)$ [subst. $\Delta = (C_u - C_d)/S_0(u-d)$]

$C(1+R) = C_d - (C_u - C_d)\, dS_0 / S_0(u-d) + (C_u - C_d)\, S_0(1+R) / S_0(u-d)$

$C(1+R) = [\, C_d\,(u-d) - (C_u - C_d)d + (C_u - C_d)(1+R)] / (u-d)$

$C(1+R) = [\, C_d\, u - C_d\, d - C_u\, d + C_d\, d + C_u\,(1+R) - C_d\,(1+R)] / (u-d)$

$C(1+R) = \{C_u\,[(1+R) - d] + C_d\,[u - (1+R)]\,\} / (u-d)$

$C(1+R) = C_u\,[(1+R) - d] /(u-d) + C_d\,[u - (1+R)] /(u-d)$

$C = [\, C_u\, p + C_d\,(1-p)\,] /(1+R)$ where $p = [(1+R) - d] /(u-d)$

$$C = [C_u p + C_d (1-p)] / (1+R) \quad \text{where } p = [(1+R) - d] / (u-d)$$

d. Interpretation:

For a risk neutral investor (be careful)*,
p is the probability that the stock will increase.

Thus, in a risk neutral world, the call value may be interpreted as the **expected payoff of the option discounted back at the riskfree rate**.

a. Be careful*: p is not precisely the same as the probability of up move.
Still, p depends on u & d, which do reflect probability of an up move.
Values of u & d must be picked to give correct E(S) & σ_S (more later).

f. Our Example:

$(1 + R) = 1.25$;

S = $50;

$u = 1.5$;

E = $50;

$d = 0.5$;

$C_u = \$25$;

$C_d = \$0$;

$p = [(1+R)-d] / (u-d) = (1.25 - .5)/(1.5 - .5)$
$\approx (1.25 - .5) = .75$;

$(1 - p) = .25$;

$$C = [\$25 (.75) + \$0 (.25)] / 1.25 = \$18.75 / 1.25$$

C. **Two-Period Binomial Model** (N=2).

1. Suppose call expires in one year (as before).
   We can extend this framework to two periods
   by splitting the year into two 6-month periods.

u^2S_0
$C_{uu} = \max\{u^2S_0\text{-}E,0\}$

uS_0
$C_u = \max\{uS\text{-}E,0\}$

S_0
(C)

$udS_0 = duS_0$
$C_{ud} = C_{du} = \max\{udS_0\text{-}E,0\}$

dS_0
$C_d = \max\{dS_0\text{-}E,0\}$

d^2S_0
$C_{dd} = \max\{d^2S_0\text{-}E,0\}$

Period 0 Period 1 Period 2

Each branch in Period 2 is like the One-Period Binomial Model:

$$C_u = [\, C_{uu}\, p + C_{ud}\,(1\text{-}p) \,] /(1+R/2)\ ;$$

$$C_d = [\, C_{ud}\, p + C_{dd}\,(1\text{-}p) \,] /(1+R/2)\ .$$

Likewise, for the first period:

$$C = [\, C_u\, p + C_d\,(1\text{-}p) \,] /(1+R/2)\ ;$$

where $(1+R/2)$ = compound interest over each 6-month period.
[e.g., if $R = 8\%$ per annum, then $R/2 = 4\%$ per 6-month period]

2. Two-Period Binomial Model Formula

   a. Each branch in Period 2 can be examined using the One-Period Binomial Model:

      $C_u = [\, C_{uu}\, p + C_{ud}\,(1\text{-}p)\,] /(1+R/2)$;

      $C_d = [\, C_{ud}\, p + C_{dd}\,(1\text{-}p)\,] /(1+R/2)$.

      Likewise, the branch in Pd 1 is:

      $C = \{\, C_u\, \mathbf{p} + C_d\, \mathbf{(1\text{-}p)} \,\} /(1+R/2)$

      where R/2 = riskfree rate earned over each 6-month period.

   b. Then we can solve for C by substituting for C_u and C_d:

$$C = \{\, [C_{uu}\, p+C_{ud}\,(1\text{-}p)]/(1+R/2)\, \mathbf{p} + [C_{ud}\, p+C_{dd}\,(1\text{-}p)]/(1+R/2)\, \mathbf{(1\text{-}p)} \,\}/ (1+R/2)$$

$$\boxed{C = \{\, C_{uu}\, p^2 + C_{ud}\, p(1\text{-}p) + C_{du}\,(1\text{-}p)p + C_{dd}\,(1\text{-}p)^2 \,\} / (1+R/2)^2}$$

   c. Simply the 1-Period Binomial Model applied twice.

      i. Interpretation is maintained:

         Call value is expected payoff over 2 periods discounted (twice) at the (1-period) riskfree rate.

   d. The terms in brackets are a *binomial expansion* of the terms within brackets in the 1-period model: $[p+(1\text{-}p)]^2$. Generalizing to N periods: $[p+(1\text{-}p)]^N$.

D. **N-Period Binomial Model**.

1. Suppose call expires in T years (the general case). Extend framework to N periods during those T years by splitting time to maturity (T) into N pds of length, T/N.

   Then the N-Period Binomial Model formula is:

   $$C = S * B[a,N,p'] - E / (1+R/N)^N * B[a,N,p]$$

   where:

   p = [(1+R/N) - d]/(u - d), and p' = [u / (1+R/N)] p;

   a = the lowest number of up moves out of N trials for which the call takes on a positive value at expiration (finishes itm).

   B[a,N,p'] = cumulative probability that the number of up moves ≥ a out of N trials, where the probability of an up move is p' (cumulative prob. that option will finish itm).

   B[a,N,p'] and B[a,N,p] can be found in the tables for the Binomial distribution, given a, N, and p or p'.

   Note: If time to maturity = T = 1 year, and N = 4 periods, then $(1+R/N)^N = (1+R/4)^4$.

2. How to set p, u, and d for the Binomial Model

The parameters, p, u, and d must be set to give correct values for $E(S_N)$ and σ_S during each time period. This choice is critical to performance of the Binomial Model. To solve for these 3 parameters, we need 3 conditions.

a. Given a risk neutral world, stock's expected return is R.

$$E(S_N) = S_0\,(1+R/N)^N = p\,uS_0 + (1-p)\,dS_0 \qquad [\div S]$$

$$(1) \qquad (1+R/N)^N = pu + (1-p)d\,.$$

This represents the ***first condition*** for p, u, and d.

b. Define: σ_S = instantaneous variance of S, over a one-year interval (annualized).

Thus, over time interval T/N, we have std dev., $\sigma_S \Delta t$.

It turns out that the variance of S over interval Δt leads to a ***second condition*** on p, u, and d:

$$(2) \qquad \sigma_S\,(T/N) = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2\,.$$

c. A ***third condition*** is also often imposed:

$$(3) \qquad u = 1/d\,.$$

If this condition is imposed, the tree "recombines;" this greatly simplifies the computations.

d. For large N, these 3 conditions are satisfied by:

$$u = e^{\sigma\ \mathrm{sqrt}(T/N)},\ d = 1/u,\ \text{and}$$

$$p = [(1+R/N)-d)]\,/(u-d)$$

E. **Black-Scholes Model**.
Limiting Case of the N-Period Binomial Model.

1. If we let $N \rightarrow \infty$, we split the time period until exp. (e.g. 1 year) into shorter and shorter intervals, until the stock price is allowed to move continuously.

   Then Binomial Model converges to Black/Scholes Model.

2. More formally, ***IF*** we make the following assumptions:

   a. Stock price (S_N) follows the lognormal distribution.
      -- S follows a random walk;
      -- S_N has lognormal distribution;
      -- $\ln(S_N)$ and $\ln(S_N/S_0)$ have normal distributions.

   b. No transactions costs or taxes;
      all securities are perfectly divisible.

   c. No dividends on stock paid during option's life.

   d. No (lasting) riskless arbitrage opportunities.

   e. Security trading is continuous.

   f. Can borrow or lend at constant riskless rate, r.

   THEN Binomial Model reduces to Black/Scholes Model:

   $$C = S_0 * N(d_1) - E e^{-RT} * N(d_2)$$

   $$P = E e^{-RT} * N(-d_2) - S_0 * N(-d_1)$$

   where $d_1 = [\ \ln(S_0/E) + (R + \sigma^2/2)T\ ] / \sigma\,(T)^{1/2}$

   and $d_2 = d_1 - \sigma\,(T)^{1/2}$.

Note: $N(d_i) = \int_{-\infty}^{d_i} f(z)\,dz$, where f(z) = std normal cdf.

$N(-\infty) = 0.0$
$N(0) = 0.5$
$N(+\infty) = 1.0$

F. Comparing Black/Scholes with N-Period Binomial Model

N-Pd Binomial Model: $C = S_0 * B[a,n,p'] - E(1+R/N)^{-N} * B[a,n,p]$

Black/Scholes Model: $C = S_0 * N(d_1) - E e^{-R T} * N(d_2)$

1. Cox, Ross, and Rubinstein (JFE, 1979) show that,
   as $N \to \infty$, $B[a,N,p'] \to N(d_1)$ and $B[a,N,p] \to N(d_2)$,
   and the two formulas converge.

   Thus, we can use same intuition as the Binomial Model.
   The B/S call value is the expected payoff of the option
   discounted back to the present at the riskless rate.

2. More Intuition:

   a. Recall, lower bound for call: $C \geq S_0 - E(1+R)^{-T}$

   b. Observe, Black Scholes weights S by $N(d_1)$ [or $N(-d_1)$]
      and weights $E(1+R)^{-T}$ by $N(d_2)$ [or $N(-d_2)$].

   c. That is, the component, S, is weighted by $N(d_1)$,
      which turns out to be the hedge ratio, Δ.
      [i.e. a riskless hedge portfolio contains
      $N(d_1)$ shares of stock for each call written.]

   d. The component, $E(1+R)^{-T}$, is weighted by $N(d_2)$,
      which is the probability the call will finish in-the-money.

G. Consider **Volatility** -- σ_S in Black/Scholes Model.

1. A measure of uncertainty about stock *returns*.
   a. Standard deviation of return on stock in 1 year, where return is continuously compounded.
   b. Often expressed as % per annum.
   c. "Old economy stocks" σ ranges from .2 to .4 (20% - 40%).
   d. "New economy stocks" σ ranges from .4 to .6 (40% - 60%).

2. As a rough approximation, $\sigma (T)^{½}$ is the std. dev. of proportional changes in S over time T.
   a. Suppose σ = 30%, and T = .5 (6 months);
   b. Then the std. dev. of returns over 6 months is
      $\sigma (T)^{½} = 30 (.5)^{½} = 21.2\%$;
   c. the std. dev. of returns in 3 months is
      $\sigma (T)^{½} = 30 (.25)^{½} = 15.0\%$; and so on.

3. The "square root effect" is important in assessing risk.
   a. In general, uncertainty about S increases as the square root of how far ahead we are looking.

4. Problem: σ is unobservable; must estimate volatility.
   Use historical data on S_i (**Historical Volatility**).

   a. Given *daily* data for 1 year on S_i (i = 1-252 trading days),

      i. Compute *daily* returns, $x_i = \ln(S_i/S_{i-1})$; i = 2-252;

      ii. Compute std dev of *daily* return = $[(1/251)\ \Sigma(x_i-x)^2\ ]^{1/2}$;

      iii. Convert to std. dev. of *annual* return, $*\ (252)^{1/2}$;

      iv. Can now use this as input in Black/Scholes.

   b. Problem: σ may not be constant for entire year.

      i. Common Practice:
         use daily closing prices from most recent 90 days.

   c. In General, define:

      N = # of observations (equally spaced - daily, weekly, …);

      S_i = stock price at end of i^{th} interval (day, week, …);

      τ = length of each time interval in years (day, week, …).

      Let $\sigma = [1/(N-2)\ \Sigma(x_i-x)^2\ ]^{1/2}$
      = std. dev. of returns over time interval, τ
      (e.g. daily, weekly, ...).

5. **Implied Volatility**.