E&G, Ch. 13: The Standard CAPM

I. Background.

Capital Market Theory makes assumptions,
& derives equilibrium relations between risk & return.

A. Portfolio Selection model.

1. Portfolio choice determined by maximizing E(U) subject to the limitations of Opportunities Locus.

2. Focus on Opportunities Locus.
   Consider N assets, with returns, R_i ; i = 1-N.

   a. The Portfolio:
   w_i, = portion of portfolio in asset i; $\Sigma w_i = 1$.

   b. No short selling allowed; $0 \leq w_i \leq 1$.

   c. Short selling allowed (w_i may be anything).

   d. Portfolio Return:
   i. $R_p = \Sigma w_i R_i$ -realized, ex post;
   ii. $E(R_p) = \Sigma w_i E(R_i)$ -expected, ex ante.

   e. Portfolio Risk:

   $\sigma_p^2 = \Sigma \Sigma w_i w_j \sigma_{ij} = \Sigma \Sigma w_i w_j (\rho_{ij} \sigma_i \sigma_j)$.

II. The CAPM.

A positive theory, applying the normative
2-parameter (mean, variance) Portfolio Model.

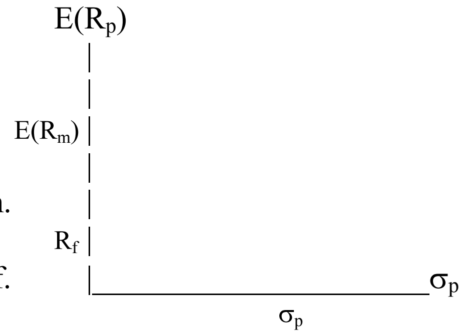
A. Assumptions:

1. No transactions costs (frictionless mkt);
2. Assets infinitely divisible;
3. No taxes;
4. Perfect competition (individual cannot affect P);
5. Investors make decisions based on Pf Model;
6. Unlimited short sales allowed;
7. Unlimited lending & borrowing @ R_f;
8. Homogeneous expectations;
9. All assets are marketable;
10. Investors are risk averse, & Max E(U) of wealth.

$\rightarrow$ Opportunities Locus is Capital Market Line (CML).

B. Implications of Assumptions, derived already.

1. Picture.



2. Separation Thm.

3. M is the mkt pf.

4. CML Relation.

   $E(R_p) = R_f + \{[E(R_m) - R_f]/\sigma_m\}\ \sigma_p$.

   Displays relation between risk & return
   for any efficient portfolio.

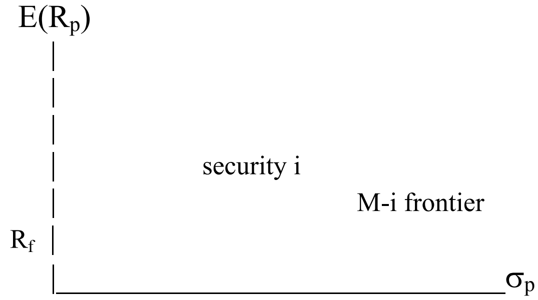
   Slope, $\{[E(R_m) - R_f]/\sigma_m\}$, is 'price of risk'
   in terms of $E(R_p)$ gained or lost.

   Displays that appropriate measure of risk
   for efficient portfolios is σ_p .

5. Note: $E(R_p)$, σ_p, $E(R_m)$, & σ_m
   are all derived from expectations about
   $E(R_i)$ & σ_i for individual securities, i = 1-N.

   The CML is an ex ante relation;
   in terms of expectations, not realizations.

C. Deriving the Security Mkt Line (SML or CAPM).



1. Consider CML, but concentrate on the relations between risk & return on all possible portfolios that are combinations of M & the ith security;
   - The M-i Frontier.
   - How is $E(R_i)$ determined relative to M?

2. **The Tangency Condition**:
   M-i Frontier must be tangent to the CML at M.
   i.e., slope of M-i Frontier = slope of CML at M.

   Proof: At M, weight on sec. i = its weight in M;
   At weights slightly > or < this weight,
   the M-i Frontier must be dominated by
   opp. locus, that is tangent to CML at M.

   Wish to calculate slope of the M-i Frontier at M. This will tell us how $E(R_i)$ varies when we add one unit of security i to the market portfolio.

3. Let z be the portfolio combining security i & M.

$$z = w_i [i] + (1-w_i)[M]$$

We are interested in how $E(R_z)$ & σ_z change,
as we consider all poss. Combin's of [i] and [M]
[as we add another unit of security i $\rightarrow$ vary w_i].

$$E(R_z) = w_i E(R_i) + (1-w_i) E(R_m)$$

$$\sigma_z^2 = w_i^2 \sigma_i^2 + (1-w_i)^2 \sigma_m^2 + 2 w_i (1-w_i) \rho_{im} \sigma_i \sigma_m$$

The slope of the M-i Frontier can be obtained by
first computing $dE(R_z)/dw_i$ and $d\sigma_z/dw_i$,
and then implementing the chain rule to get:

$$dE(R_z)/d\sigma_z = [dE(R_z)/dw_i] / [d\sigma_z /dw_i].$$

This slope will describe the relation between
risk (σ_z) and expected return ($E(R_z)$),
as we add another unit of security i to the Market.

(Next page.)

3. (cont.) $E(R_z) = w_i E(R_i) + (1-w_i) E(R_m)$

$$\sigma_z^2 = w_i^2 \sigma_i^2 + (1-w_i)^2 \sigma_m^2 + 2 w_i (1-w_i) \rho_{im} \sigma_i \sigma_m$$

a. First, $dE(R_z)/dw_i = E(R_i) - E(R_m)$

b. Second, $\sigma_z = [w_i^2 \sigma_i^2 + (1-w_i)^2 \sigma_m^2 + 2w_i (1-w_i)\sigma_{im}]^{1/2}$;

$$\rightarrow d\sigma_z /dw_i = ½ (\sigma_z^2)^{-½} [2w_i \sigma_i^2 + 2(1-w_i)(-1)\sigma_m^2 + 2\sigma_{im} - 4w_i \sigma_{im}]$$
$$= 1/\sigma_z [w_i \sigma_i^2 + (w_i-1) \sigma_m^2 + \sigma_{im} - 2w_i \sigma_{im}]$$
$$= 1/\sigma_z [w_i (\sigma_i^2 + \sigma_m^2 - 2\sigma_{im}) + (\sigma_{im} - \sigma_m^2)]$$

c. Then, by the Chain Rule:

$$\frac{dE(R_z)}{d\sigma_z} = \frac{E(R_i) - E(R_m)}{1/\sigma_z [w_i (\sigma_i^2 + \sigma_m^2 - 2\sigma_{im}) + (\sigma_{im} - \sigma_m^2)]}$$

(the slope of the M-i Frontier in $[E(R_z),\sigma_z]$ space).

d. Thus, the slope of the M-i Frontier ***at M*** (where $w_i = 0$) is:

$$\frac{dE(R_z)}{d\sigma_z} = \frac{E(R_i) - E(R_m)}{1/\sigma_z (\sigma_{im} - \sigma_m^2)} = \frac{\sigma_z [E(R_i) - E(R_m)]}{(\sigma_{im} - \sigma_m^2)}$$

↓

(But at $w_i = 0$, $\sigma_z = \sigma_m$); $= \dfrac{\sigma_m [E(R_i) - E(R_m)]}{(\sigma_{im} - \sigma_m^2)}$

e. We also know that, ***at M***,
this slope = slope of CML = $[E(R_m) - R_f] / \sigma_m$.
i.e., the Tangency Condition implies:

$$\frac{E(R_m) - R_f}{\sigma_m} = \frac{[E(R_i) - E(R_m)]\sigma_m}{(\sigma_{im} - \sigma_m^2)}$$

4. The **Tangency Condition implies**:

$$\frac{E(R_m) - R_f}{\sigma_m} = \frac{[E(R_i) - E(R_m)]\,\sigma_m}{(\sigma_{im} - \sigma_m^2)}$$

Solve this for $E(R_i)$:

$$E(R_i) - E(R_m) = [E(R_m) - R_f]\,(1/\sigma_m^2)\,(\sigma_{im} - \sigma_m^2)$$

$$E(R_i) - E(R_m) = [E(R_m) - R_f]\,(\sigma_{im}/\sigma_m^2) - [E(R_m) - R_f]$$

$$E(R_i) = R_f + [E(R_m) - R_f]\,(\sigma_{im}/\sigma_m^2)$$

Or: $$E(R_i) = R_f + [E(R_m) - R_f]\,\beta_{im}$$

This expresses a security's $E(R_i)$ in terms of its β_{im}.

- The Security Market Line (SML); or the CAPM.

D. Implications of Security Market Line (SML).

$$E(R_i) = R_f + [E(R_m) - R_f]\, \beta_{im}$$

1. β_{im} is relevant measure of risk for any security.

2. If $\beta_{im} = 1$, $E(R_i) = E(R_m)$.
   Same risk as market, same expected return.

3. If $\beta_{im} > 1$, $E(R_i) > E(R_m)$.
   More risk than market, more expected return.

4. If $\beta_{im} < 1$, $E(R_i) < E(R_m)$.
   Less risk than market, less expected return.

5. If $\beta_{im} = 0$, $E(R_i) = R_f$.
   No risk, expected return = riskfree rate.

6. These are *ex ante*; expectations, not realizations.

7. These are *equilibrium* relations, not realizations.

   High β stocks have high $E(R_i)$ [$> E(R_m)$],
   but may have low realized returns (R_i) some time!
   CAPM cannot guarantee you'll beat Mkt!

8. SML applies to any security or inefficient pf.
   → Every security lies on the SML.
   → Every possible portfolio also does,
   since $E(Rp) = \Sigma w_i\, E(R_i)$ & $\beta_p = \Sigma w_i\, \beta_{im}$.
   → (CML only applies to efficient portfolios.)

9. If 2 securities have same β_{im}, then same $E(R_i)$.

10. Security's std deviation (σ_i) is not the only factor
    that affects its $E(R_i)$; covariance with market does!

11. S-I Market Model: $R_i = \alpha_i + \beta_{im}R_m + \varepsilon_i$
    Result 1: $E(Ri) = \alpha_i + \beta_{im}E(R_m)$
    Result 2: $\sigma_i^2 = \beta_{im}^2\,\sigma_m^2 + \sigma_{\varepsilon i}^2$
    → CAPM implies only systematic risk is priced.

III. Applying the CAPM to measure and manage risk.

A. How would you measure the market risk of your portfolio?

1. If one security, market risk = β_{im} ;
2. If portfolio p, market risk = β_{pm} = $\Sigma X_i \beta_{im}$.

B. Suppose you expect market to drop 10% soon.
How would you eliminate the market risk of your portfolio?
Your portfolio has risk = $\beta_{pm} > 0$; You want $\beta_{pm} = 0$;

1. Sell all your stocks → cash. Then $\beta_{pm} = 0$.

   Problem: High trans. costs (TC); May want $\beta_{pm} > 0$ soon.

2. Hedge: Short a stock whose $\beta_{im} = \beta_{pm}$.
   Receive proceeds now, must buy back later;
   expect mkt to drop, expect to owe less later.

		E(ΔWealth)		Risk (β_{im})
Own pf with	$\beta_{im} = \beta_{pm}$;	(-10%)xβ_{pm}	[own less]	+ β_{pm}
Short stock w/	$\beta_{im} = \beta_{pm}$;	(+10%)xβ_{pm}	[owe less]	- β_{pm} .
		0		0

Problem with 2 : Hedge (single stock whose $\beta_{im} = \beta_{pm}$) has $\sigma_{\varepsilon i}^2$.
E(ΔWealth) ignores unsystematic risk in the single stock.

3. Hedge: Short a diversified portfolio whose $\beta_{im} = \beta_{pm}$.
   Then unsystematic risk is gone, and you are sure risk = 0.

C. Elaborate on B.2. versus B.3. :

1. $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$

2. $R_p = \Sigma X_i \alpha_i + \Sigma X_i \beta_i R_m + \Sigma X_i \varepsilon_i$

$= \alpha_p + \beta_p R_m + \varepsilon_p$

On average, the +'s and –'s in ε_i cancel out in ε_p;
Also, they are multiplied by fractions, X_i, so that
unsystematic risk ($\sigma_{\varepsilon p}^2$) gets smaller as N increases.

3. To get unsystematic risk close to zero, need large N.

   a. Must pay large TC to get effective hedge pf, p.

   b. Can use fewer stocks in portfolio, to reduce T.C.,
   But ($\sigma_{\varepsilon p}^2$) will be larger – hedge less effective.
   Tradeoff: TC versus ($\sigma_{\varepsilon p}^2$).

   c. OR can trade securities that mimic benchmark pf's
   like SPDRs, WEBs, HOLDRs, DIAMONDS, etc.

D. Traded Securities that Mimic Benchmark Portfolios.

1. Exchanges offer tradable stock index securities.

2. AMEX Index Shares - track > 30 different benchmarks.
   a. Represent ownership in a trust that holds a portfolio that closely tracks performance and dividend yield of the benchmark index in question.

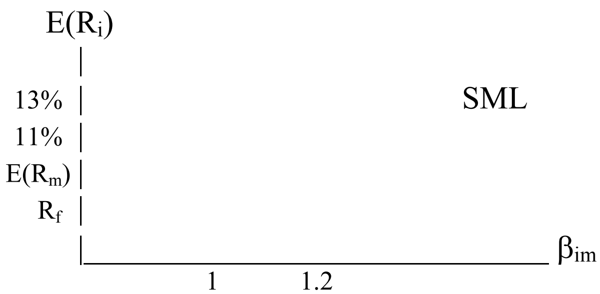
3. Most popular:
   a. SPDR's - S&P Depository Receipts, track S&P 500.
   b. Select Sector SPDR's - based on sub-indexes of S&P.
   c. DIAMONDS - based on the Dow.
   d. Nasdaq 100 Index Tracking Stock (ticker QQQ).
   e. WEBS - World Equity Benchmark Shares; based on Morgan Stanley's country indexes.
   f. Internet HOLDRs (HHH) based on 20 internet stocks.

4. Example – SPDRs.
   a. Trust holds pf of stocks that closely tracks the index.
   b. SPDR entitles holder to receive qtrly cash dividend corresponding to div. of stocks in trust less expenses.
   c. SPDR expenses ≈ 0.185% (.00185 of amt invested).

IV. Arbitrage activity forces CAPM to hold.

A. SML is linear relation: $E(R_i) = R_f + [E(R_m) - R_f]\,\beta_{im}$

Given the security's expected β_{im}, its $E(R_i)$ lies on this line.



2 securities with same β_{im} should have same $E(R_i)$ in equilibrium.

1. Reflects ex ante *equilibrium* relation, based on expectations.

2. What if security's combination $\{E(R_i), \beta_{im}\}$ lies above SML?

   a. Graph shows 3 securities, A, B, & C, lie on SML.
   b. Security C has $\beta = 1.2$; $E(R_C) = 11\%$.
   c. Security D has $\beta = 1.2$; $E(R_D) = 13\%$.
      Security D has same risk as C, but higher $E(R_i)$.

3. Arbitrage opportunity:

	ΔWealth today	E(ΔWealth).	Risk (β_{im})
Short C	+$100	owe $111	-1.2
Buy D	- $100	worth $113	+1.2
	$0	+ $2	0

Portfolio uses no wealth, has no risk → arbitrage portfolio.
In equilibrium, should pay $E(R) = 0$. Here $E(R) > 0$.

Investors will do this, buying D until they bid up its price, and bid down its $E(R_D)$ to 11%, warranted by its $\beta = 1.2$.

B. Application to Capital Budgeting.

1. In practice, finance theory suggests a manager should invest in a project (or security) if its NPV > 0.
   What is appropriate discount rate?
   Should be based on project's (or security's) risk, β_{im} .

2. The CAPM suggests a manager should calculate β_i to determine $E(R_i)$ for each alternative investment.
   $E(R_i)$ is appropriate discount rate, adjusted for risk.

3. The SML can be used directly as a criterion for acceptance or rejection of alternative investments.

4. Consider project as potential security to be held.
   If project lies above SML, accept; below, reject.

5. In more traditional terms, cost of capital for a project is expected rate of return, $E(R_i)$, shown by the SML for all projects with equal risk, β_i.

   Thus, expected $ cash flow from project considered should be discounted at this rate of interest;
   if NPV > 0, accept; < 0, reject.

$E(R_i)$

$E(R_I)$ |

k_j |

$E(R_{II})$ |

R_f |______________________________ β_{im}

β_{II} β_k β_I

Project I:
$E(R_I) > k_j$
but < SML
(high risk!)

Project II:
$E(R_{II}) < k_j$
but > SML
(low risk!)

C. Application to Valuing Stocks.

The Wells Fargo Valuation Model (E&G, Ch. 18).

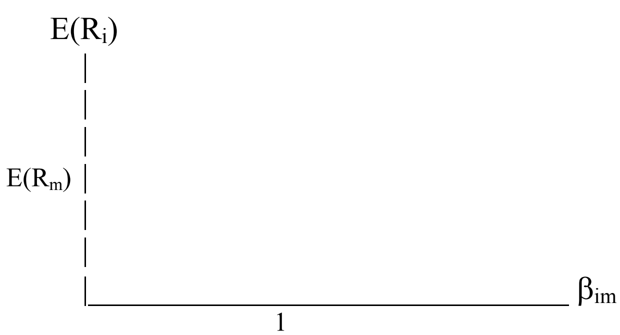
1) Use DCF Model to predict k_i ;
2) Obtain estimates of risk, β_i;
3) Plot firms' (β_i , k_i) forecasts;
4) Fit implied SML;
5) Accept if above SML, reject if below.

Combines theoretically sound 3-Period DCF Model
and cross-sectional regression analysis of SML relation.

1. First uses DCF Model to generate expected return (k_i)
   implicit in the current price of the i^{th} stock,
   given assumptions about firm growth in 3-pd model.

2. Then combines expected return (k_i) with estimated β_i
   for different companies; plot, fit SML;
   above SML→overvalued; below SML→undervalued.

This constructed SML is an expectational construct;
Represents relation between *expected* return *expected* risk.
Most approaches to estimate the SML use
actual historical data, not forecasts of *expected* (k_i, β_i).



D. Prices and the CAPM.
Or, The CAPM as a Valuation Model.

Assume: One-Period Model (built into CAPM)

1. Want to value an asset that has a risky payoff.

   P_{j0} = current theoretical value of risky asset (appropriate price today; *want to find this*);

   P_{je} = risky end-of-period payoff realized;

   $(P_{je}-P_{j0})$ = capital gain + dividend;

   $R_j = (P_{je}-P_{j0})/P_{j0}$ = actual return to be earned, ex post, on risky asset;

   $E(R_j) = [E(P_{je}) - P_{j0}]/ P_{j0}$ = expected return.

   CAPM: $E(R_j) = R_f + \{[E(R_m) - r_f]/\sigma_m^2\}\ \sigma_{jm}$

   Or: $E(R_j) = R_f + \lambda\, \sigma_{jm}$;

   where $\lambda = \{[E(R_m) - r_f]/\sigma_m^2\}$ = price of risk.

   Thus: $[E(P_{je}) – P_{j0}]/ P_{j0} = R_f + \lambda\, \sigma_{jm}$.

   Solve for P_{j0}: $E(P_{je}) - P_{j0} = P_{j0} R_f + P_{j0}\, \lambda\, \sigma_{jm}$;

   Or: $E(P_{je}) = P_{j0} + P_{j0} R_f + P_{j0}\, \lambda\, \sigma_{jm}$;

   Or: $P_{j0} = E(P_{je}) / [1 + (R_f + \lambda\, \sigma_{jm})]$;

   Or: $P_{j0} = E(P_{je}) / [1 + E(R_j)]$.

"Risk-Adjusted Rate-of-Return Valuation Formula"

"Risk-Adjusted Rate-of-Return Valuation Model"

$$P_{j0} = E(P_{je}) / [1 + R_f + \lambda\, \sigma_{jm}];$$

1. An equivalent approach:
   deduct a risk premium from $E(P_{je})$ in numerator;
   then discount this "certainty equivalent" @ R_f.

First, rewrite σ_{jm}:

$$\sigma_{jm} = Cov(R_j, R_m) = Cov\ \{[(P_{je}-P_{j0})/P_{j0}], R_m\}$$

$$= E\ \{[(P_{je}-P_{j0})/P_{j0} - (E(P_{je})-P_{j0})/P_{j0}], R_m\}$$

$$= (1/P_{j0})\ Cov(P_{je}, R_m).$$

Then, substitute this into the expression above:

$$P_{j0} = E(P_{je}) / [1 + R_f + \lambda\, \sigma_{jm}];$$

$$P_{j0} = E(P_{je}) / [1 + R_f + \lambda\, (1/P_{j0})\ Cov(P_{je}, R_m)];$$

Solving for P_{j0}:

$$P_{j0}\, [1 + R_f + \lambda(1/P_{j0})\ Cov(P_{je}, R_m)] = E(P_{je})$$

$$P_{j0}\, (1+R_f) = E(P_{je}) - \lambda\ Cov(P_{je}, R_m)$$

$$P_{j0} = \{E(P_{je}) - \lambda\ Cov(P_{je}, R_m)\} / (1+R_f).$$

Called "Certainty Equivalent Valuation Formula."