E&G, Ch. 8: Multi-Index Models & Grouping Techniques

I. Multi-Index Models.

A. The General Multi-Index Model:

$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots + b_{iL}I_L + c_i$

Explanation:

1. Let $I_1 = R_m$; I_2 = interest rate; I_3 = industry; …
2. b_{i1} is analogous to β_i in Single-Index Model; likewise for other b_{ik}.
3. a_i = intercept; part of return, R_i, unique to firm i; this is *nonrandom*.
4. c_i = error term; part of return, R_i, unique to firm i; this is *random* variation in R_i not due to any I_j.

By Definition:

1. σ_{ci}^2 = residual variance of stock i (for stock i = 1-N);
2. $\sigma_{Ij}^2 = E(I_j-\bar{I}_j)^2$ = variance of index j (for index j = 1-L).

By Construction:

1. mean of $c_i = E(c_i) = 0$; for any firm i = 1-N;
2. covariance between indexes $= E[(I_j-\bar{I}_j)(I_k-\bar{I}_k)] = 0$; for any pair of indexes, $j \neq k$; j, k = 1-L.
3. covariance of residuals for stock i & index j; $E[c_i(I_j-\bar{I}_j)] = 0$; for any firm, i = 1-N, and any index, j = 1-L.

By Assumption:

1. $E(c_i c_j) = 0$; for any i, j = 1-N.

Consider 'By Construction – 2.'

2. covariance between indices $= E[(I_j-\bar{I}_j)(I_k-\bar{I}_k)] = 0$;
   for any pair of firms, $i \neq j$;

Implies different indexes (factors) are uncorrelated.

Unrealistic?
If we used *actual* R_m and interest rates, $Cov(I_1,I_2) \neq 0$.

Fact: Can always construct a new set of indexes, I_j,
from the original set of indexes (say, I_j^*)
that are uncorrelated with each other.

(See Appendix A.)

Don't have to do this; could use *actual* I_j^*.
However, doing this greatly simplifies:

a. computation of risk; and
b. selection of optimal portfolios.

Expressions for $E(R_i)$, σ_i^2, and σ_{ij} (See Appendix B).

Expected Return:

$R_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 + \ldots + b_{iL}\bar{I}_L$ (Result 1)

Variance of Return:

$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \ldots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ci}^2$ (Result 2)

Covariance of Returns for securities i and j:

$\sigma_{ij} = b_{i1}\, b_{j1}\, \sigma_{I1}^2 + b_{i2}\, b_{j2}\, \sigma_{I2}^2 + \ldots + b_{iL}\, b_{jL}\, \sigma_{IL}^2$ (Result 3)

Can estimate $E(R_i)$, σ_i^2, & σ_{ij} if we have estimates of:

(N) a_i; $i = 1\text{-}N$;
(NL) b_{ik}; $i = 1\text{-}N$ stocks, $k = 1\text{-}L$ indexes;
(N) σ_{ci}^2; $i = 1\text{-}N$ stocks;
(L) $E(I_k)$; $k = 1\text{-}L$ indexes;
(L) σ_{Ik}^2; $k = 1\text{-}L$ indexes.

Requires $(2N + 2L + LN)$ inputs.

B. The Industry Index Model.

$R_i = a_i + b_{im}R_m + b_{i1}I_1 + b_{i2}I_2 + \ldots + b_{iL}I_L + c_i$

where R_m = market index;
I_j = indexes for industries (j = 1-L);
with all indexes constructed to be uncorrelated.

1. Suppose firm i operates mainly in one industry (j);
   Assume $b_{ik} = 0$ for other industries ($k \neq j$):

   $$R_i = a_i + b_{im}R_m + b_{ij}I_j + c_i$$

   for the i^{th} firm in industry j (j = industries 1-L).

   Again, each industry index is constructed to be
   be uncorrelated with R_m and the other I_k ($k \neq j$).

2. This assumption implies the same 2-Index Model
   for all firms *within* a given industry, and
   a different 2-Index Model for firms in every other industry.

3. Simplifies correlation structure across firms (i & k):

   For firms in same industry (j), $\sigma_{ik} = b_{im}b_{km}\sigma_m^2 + b_{ij}b_{kj}\sigma_{Ij}^2$;

   For firms in different industries, $\sigma_{ik} = b_{im}b_{km}\sigma_m^2$.

   Note: If firms in same industry, 2 sources of covariance:
   common ρ with R_m, and common ρ with industry (I_j).

   If 2 firms in different industries, only 1 source:
   common ρ with R_m.

4. Inputs required:

   (N) a_i; $i = 1$-N;
   (N) b_{im}; $i = 1$-N stocks;
   (N) b_{ij}; $i = 1$-N stocks in industry j;
   (N) σ_{ci}^2; $i = 1$-N stocks;
   (2) $E(R_m)$ & σ_m^2;
   (L) $E(I_j)$; $j = 1$-L indexes (to compute $E(R_i)$ & σ_{Ij}^2);
   (L) σ_{Ik}^2; $k = 1$-L indexes.

   Requires (4N + 2L + 2) inputs. (< N(N-1)/2 !)

C. How Well Do Multi-Index Models Work?
(Recall: S-I Model forecasts ρ better than hist. ρ matrix.)

1. Multi-Index Models perform somewhere in between Single-Index Models and Historical Correlation matrix.

   a. As more indexes are added, more complex, and *historical* correlation matrix is more closely reproduced.

      However, this does *not* imply that *future* correlation matrices will be better forecast!

   b. Tests (2 kinds):

      (1) Statistical Significance –
      Is there improvement in forecasts (vs actuals) that is statistically significant?

      (2) Economic Significance –
      Is there difference in return or profit to be made using one forecast technique versus another?

      - Examine future returns from selecting portfolios based on each technique.

      - From estimates of ρ_{ij} matrix, determine opportunity locus & choose optimal pf's.

      - Does one set of estimates of ρ_{ij} matrix lead to opportunity locus & pf's that perform better? (i.e., higher $E(R_p)$ at various levels of σ_p^2).

2. **Elton & Gruber study**. Found:

Single-Index Model did better than Multi-Index Model in both (1) & (2).

Adding indexes (beyond R_m) got higher R^2;
i.e., better explanation of *historical* ρ's.

However:
(1) led to poorer prediction of *future* ρ's;
(2) led to selection of pf's that tended to have lower returns at each risk level.

Implication:
The added indexes apparently introduced more random noise than useful info for forecasting.

3. **Cohen & Pogue Study**.

Concentrated on (2) – tests of economic significance.
Divided stocks into industries using SIC codes.
For each industry grouping, then ran:
- Single-Index Models (with R_m)
- Multi-Index Models (with R_m and one I_j).

Found (like E&G study):
- Multi-Index Models got higher R^2.
- BUT Single-Index Model had better properties; led to lower expected risks, & was simpler to use.

4. Comment: SIC code classifications may be bad.

   a. there are many multi-product firms;
      SIC classification is arbitrary for such firms.

   b. Companies in same 'industry' may have
      different operating & risk characteristics.
      (Not the best grouping technique).

5. Another E&G study, & Farrell study.

   a. Considered Industry-Index Model again.
      Instead of SIC classifications,
      formed their own 'industry groupings'
      according to tendency of firms to act alike.

   b. Formed *homogeneous groups* of firms;
      i. After removing ρ explained by R_m,
         examine ρ's across residuals.
      ii. Stocks with highly correlated residuals
          grouped into '*pseudo-industry*,' called I_j.
      iii. Indexes for each 'pseudo-industry' (I_j)
           then used in Multi-Index Model.

   c. Found 4 major 'pseudo-industry' groupings:
      i. growth stocks;
      ii. cyclical stocks;
      iii. stable stocks;
      iv. oil stocks.

Farrell study estimated 2-Index Model (R_m & I_j)
for each pseudo-industry.

Results:

1. R_m explained 30% of variation in R_I;
2. I_j accounted for another 15%.

Not surprising; include more variables,
explain more of *past* behavior.

Relevant question is how do pseudo-industries:

1. improve forecasts of ρ_{ij},
2. select portfolios that eventually perform better?

Farrell investigated (2) – economic significance;
2-Index Model based on pseudo-industries lead to:

1. better performance than Single-Index Models at some risk levels;
2. worse performance at other risk levels.
3. On the whole, Farrell concludes his Multi-Index Model better than Single-Index Model.

Caution:

1. Dominance found by Farrell is not complete.
2. Based on one sample of stocks, one time period.
3. A different study (by Fertuck) found that pseudo-industries explained less than SIC groups.

II. Grouping Techniques or Avg Correlation Models.

Two types, based on extent of aggregation.

A. Overall Mean Model. (most aggregated)

1. Given R_{it}; i = 1, …, N stocks; t = 1, …, T periods.
   Compute historical correlation matrix (NxN)
   (pairwise ρ_{ij}'s for all pairs of stocks, i, j = 1, …, N;
   Then average all (N(N-1)/2) pairwise ρ_{ij}'s; ρ_{ij}.

   a. Then $\bar{\rho}_{ij}$ (avg across all pairwise correlations)
      is the forecast for each pairwise correlaion, ρ_{ij}.

B. Traditional Mean Model. (disaggregated)

1. Assumes there is a common mean correlation
   within & between groups of stocks
   (such as traditional or pseudo-industry groupings).

2. Examples:

   a. avg $\bar{\rho}_{ij}$ for all i, j within steel ind. (steel $\bar{\rho}_{ij}$).

   b. avg $\bar{\rho}_{ij}$ for all i in steel & all j in chemical ind.

      (steel/chemical $\bar{\rho}_{ij}$).

C. Comparison of Performance.

Overall Mean Model has been extensively tested
against Single- and Multi-Index Models.
(1) Statistical Significance of forecast improvement;
(2) Economic Significance of stock selection perf.

1. Overall Mean Model (most aggregated).

   a. Outperforms Single-, Multi-Index Model, and Historical Correlation Matrix in both (1) & (2).

   b. For (1), improvements in forecast performance almost always signif. @ 5% level.

   c. For (2), big improvement achieved in performance at most risk levels.
   (↑ed return 25% @ some risk levels!)

2. Traditional Mean Model (disaggregated).
   a. Based on SIC codes.
   b. Based on pseudo-industries.

   Again, Outperforms Single-, Multi-Index Models, and Historical Correlation Matrix in both (1) & (2).

   However, ranking of 1., 2.a., and 2.b. varies for different risk levels in same time period, and for different time periods.

   Thus, 2. not always superior to 1.; need more work.

D. Mixed Models.

1. Combine attributes of Single-Index Model, Multi-Index Model, & Averaging Techniques.

   Begin with Single-Index Model
   to account for covariance due to the Market.

   Then construct a second model to explain
   the 'extra-market' covariance.

   a. S-I Model estimates covariance with R_m.

      E&G, Ch. 7, assumed extra-mkt covariance = 0.
      E&G, Ch. 8, focuses on other indexes that
      help explain more of variation in R_i.

      i.e., Multi-Index Models include other indexes
      of extra-mkt covariance.

   b. Mixed Models suggest different way to do this.
      Look at residuals of Single-Index Model;
      Compute ρ_{ij}'s from S-I Model residuals.

2. **Rosenberg study** (discussed in E&G, Ch. 7).
   Worked with fundamental β's;
   related β's to set of firm fundamental data.

   Here Rosenberg suggests computing the
   extra-mkt covariance matrix of ρ_{ij}'s
   after removing the mkt's influence
   i.e., compute ρ_{ij}'s from S-I Model residuals.

   Then regress these ρ_{ij}'s (extra-mkt covariance)
   on (114) firm fundamental variables – forecast ρ_{ij}.

   Results promising – more work needed.

3. Combine S-I model and Averaging Techniques.

   a. Instead of averaging raw ρ_{ij},
      average the ρ_{ij} computed from
      residuals of the S-I Model
      (i.e., the extra-mkt correlations).

   b. Could do all stocks (i.e., Overall Mean Model);

      or across firms within traditional industries or
      pseudo-industries, as well as across such groups
      (i.e., Traditional Mean Model).

   c. Then ρ_{ij} could be predicted by combining
      the predicted ρ_{ij} from the S-I Model with the
      extra-mkt ρ_{ij} predicted from the avging model.

      More work needed.