E&G, Ch. 7: The Single-Index Model

Implementing Portfolio Theory -
Estimating the Correlation Structure of Security Returns.

I. Motivation.

Recall: $E(R_p) = \sum_{i=1}^{N} X_i E(R_i)$

$$\sigma^2(R_p) = \sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} X_i X_j \sigma_i \sigma_j \rho_{ij}$$

A. To Use Portfolio Analysis, need following inputs.

$E(R_i)$;	$i = 1, \ldots, N$;	(N estimates required)
σ_i^2;	$i = 1, \ldots, N$;	(N estimates required)
ρ_{ij};	$i, j = 1, \ldots, N$.	(N(N-1)/2 estimates)

1. Need to forecast the correlation matrix.
   If N = 150 stocks, N(N-1)/2 = 11,175!

B. Estimation requirements for ρ_{ij} present a problem.

1. Organizational problems.

   a. Investment firms organize their analysts along industry categories;
   Analyst familiar with chemical stocks may not know correlation with steel stocks, …

2. Data problems.

   a. Large numbers of ρ's to estimate.

C. Want to Simplify Computational Needs.

1. Problems have motivated development of models to describe & predict ρ_{ij}.

   a. Models fall into two categories:
      i. Index Models. - E&G, Chapter. 7
      ii. Averaging Techniques. - E&G, Chapter. 8

2. Assume covariation among stocks due to a single common influence (or index, the market),
   - the Single-Index Model.

3. Single-Index Model is used to estimate ρ_{ij};
   to investigate efficient mkt tests & equilibrium tests;
   to learn more about how capital markets work.

II. Single-Index Models.

A. Overview.

1. Fact: When mkt ↑, most stock prices ↑.
   When mkt ↓, most stock prices ↓.

2. One reason stock prices are correlated is common response to market changes.

3. Recall, $\sigma_p^2 = (1/N)\,\bar{\sigma}^2 + (N-1)/N\,\bar{\sigma}_{ij}$.

4. A measure of ρ_{ij} can be obtained by relating stock's return to return on Market Index.

B. The Single-Index Model.

Basic Equation:	$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$;	$i = 1$ - N
By Construction:	$E(\varepsilon_i) = 0$;	$i = 1$ - N.
By Assumption:	$E[\varepsilon_i (R_m - \bar{R}_m)] = 0$;	$i = 1$ - N.
	$E(\varepsilon_i \varepsilon_j) = 0$;	$i \neq j$.
By Definition:	$Var(\varepsilon_i) = \sigma_{\varepsilon i}^2$;	$i = 1$ - N.
	$Var(R_m) = E(R_m - \bar{R}_m)^2 = \sigma_m^2$.	

C. Elaborate on The Single-Index Model

1. Definitions:

   R_i = return on stock i;

   R_m = return on Market Index;

   β_i = coefficient of R_m:
   if > 1, stock more risky than mkt;
   if < 1, stock less risky than mkt;
   -- Measures responsiveness of R_i to R_m;
   -- Part of movements in R_i related to R_m.

   α_i = intercept:
   R_i may be higher or lower than $(\beta_i R_m)$,
   depending on company characteristics;
   -- *Nonrandom* part of movements in R_i
   that are unique to firm i;
   -- If $R_m = 0$, $E(R_i) = \alpha_i$.

   ε_i = random error or 'noise' for firm i;
   R_i may not $= [\alpha_i + \beta_i R_m]$ at all times,
   for reasons specific to the firm.
   -- *Random* part of movements in R_i
   that are unique to firm i.

2. Assumptions:

   $E(\varepsilon_i) = 0$; randomness $\neq 0$, but on avg washes out.

   $E[\varepsilon_i (R_m - E(R_m))] = 0$;
   firm's randomness uncorrelated with R_m;
   If $R_m\uparrow$, ε_i should not $\uparrow$, & visa versa.

   $E(\varepsilon_i \varepsilon_j) = 0$; randomness in firm **i** uncorrelated
   with randomness in firm **j**;
   The only reason R_i & R_j move together is R_m.

   $Var(\varepsilon_i) = \sigma_{\varepsilon i}^2$ = amount of randomness about line.

3. Implications:

   If $R_m\uparrow$, expect $R_i\uparrow$ by $(\beta_i R_m)$, & $R_j\uparrow$ by $(\beta_j R_m)$;
   Thus, R_i & R_j correlated according to β_i, β_j, & ΔR_m.

   However, if $\varepsilon_i\uparrow$, do not expect R_m or ε_j to $\uparrow$.

   No effects beyond the market that account for ρ_{ij}.

4. Observations:

   ε_i is random variable [distributed $N\{0, \sigma_{\varepsilon i}^2\}$].

   R_m is random variable [distributed $N\{E(R_m), \sigma_m^2\}$].

   R_i is random variable; depends on R_m and ε_i!

   $E(R_i) = \alpha_i + \beta_i \bar{R}_m$ (Result 1).

   σ_i^2 must also depend on σ_m^2 and $\sigma_{\varepsilon i}^2$ (Result 2).

The Single Index Model, cont. $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$;

Fact:

$\beta_i = \beta_{im} = \rho_{im} (\sigma_i / \sigma_m) = [\sigma_{im} / \sigma_i \sigma_m] (\sigma_i / \sigma_m) = \sigma_{im} / \sigma_m^2$.

So, $\beta_i = \rho_{im} (\sigma_i / \sigma_m) = \sigma_{im} / \sigma_m^2$ and $\rho_{im} = \beta_i (\sigma_m / \sigma_i)$.

D. Result 1.

The expected return on a security is:

$$
\begin{aligned}
E(R_i) &= E[\alpha_i + \beta_i R_m + \varepsilon_i] \\
&= E(\alpha_i) + \beta_i E(R_m) + E(\varepsilon_i) \\
&= \alpha_i + \beta_i E(R_m);
\end{aligned}
$$

Or $$E(R_i) = \alpha_i + \beta_i \bar{R}_m ;$$

Or, $$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m .$$

E. Result 2.

The variance of return on any security is:

$$\sigma^2(R_i) = \sigma_i^2 = E(R_i - \bar{R}_i)^2 \qquad [\neq \sigma_{\varepsilon i}^2]$$

Substituting for R_i and $\bar{R}_i$:

$$\sigma_i^2 = E[(\alpha_i + \beta_i R_m + \varepsilon_i) - (\alpha_i + \beta_i \bar{R}_m)]^2$$

$$= E[\beta_i (R_m - \bar{R}_m) + \varepsilon_i]^2$$

$$= \beta_i^2 E(R_m - \bar{R}_m)^2 + 2\beta_i E[\varepsilon_i (R_i - \bar{R}_m)] + E(\varepsilon_i^2)$$

Or, $\quad \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2$

F. Result 3.

The covariance between any two securities is:

$$\sigma_{ij} = E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)]$$

$$= E\{[\alpha_i+\beta_i R_m+\varepsilon_i] - [\alpha_i+\beta_i \bar{R}_m]\} \{[\alpha_j+\beta_j R_m+\varepsilon_j] - [\alpha_j+\beta_j \bar{R}_m]\}$$

$$= E[\beta_i (R_m - \bar{R}_m) + \varepsilon_i][\beta_j (R_m - \bar{R}_m) + \varepsilon_j]$$

$$= \beta_i \beta_j E(R_m - \bar{R}_m)^2 + \beta_j E[\varepsilon_i (R_m - \bar{R}_m)] +$$

$$+ \beta_i E[\varepsilon_j (R_m - \bar{R}_m)] + E(\varepsilon_i \varepsilon_j)$$

Or, $\quad \sigma_{ij} = \beta_i \beta_j \sigma_m^2$

G. Implications.

If this specified relation (the Single-Index Model) represents the joint movement among securities, we can derive expressions for $E(R_i)$, σ_i^2, and σ_{ij}.

These are the input requirements for portfolio analysis.

Result 1: $E(R_i) = \alpha_i + \beta_i E(R_m)$ -- [E&G, $\bar{R}_i$, $\bar{R}_m$]

Result 2: $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2$

Result 3: $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$

The individual security's return is now described in relation to Market's return: $\{E(R_m),\ \sigma_m^2,\ \text{and}\ \beta_i\}$.

This allows great simplification in computing the expected return & variance of any portfolio: $\{E(R_p)\ \text{and}\ \sigma_p^2\}$.

III. Expected Return and Variance of a Portfolio.

$$E(R_p) = \sum_{i=1}^{N} X_i E(R_i) \qquad \text{[Result 1]}$$

$$E(R_p) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i E(R_m)$$

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij} \qquad \text{[Results 2 \& 3]}$$

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

If we have estimates of:

1. α_i; $i = 1 - N$; [or $E(R_i)$]
2. β_i; $i = 1 - N$; for each stock
3. $\sigma_{\varepsilon i}^2$; $i = 1 - N$; [or σ_i^2]
4. σ_m^2;
5. $E(R_m)$ for the Market

Then we can estimate $E(R_p)$ and σ_p^2.
This requires $3N+2$ estimates. [< N(N-1)/2!]

Note: Unnecessary to estimate $N(N-1)/2$ ρ's directly.
Only need to estimate the β_i's showing how each security moves with respect to the Market.

A firm organized with analysts focusing on specific groups of stocks can easily expect these inputs.

Can also employ this model (to estimate $E(R_p)$ & σ_p^2) if analysts supply estimates of:

1. $E(R_i)$; $i = 1\text{-}N$; [replaces α_i]
2. β_i ; $i = 1\text{-}N$;
3. σ_i^2; $i = 1\text{-}N$; [Replaces $\sigma_{\varepsilon i}^2$]
4. σ_m^2;
5. $E(R_m)$.

IV. Characteristics of the Single-Index Model.

A. Portfolio β_p and α_p.

$$E(R_p) = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i E(R_m)$$

If we let the portfolio's beta = $\beta_p = \Sigma X_i \beta_i$,

and the portfolio's alpha = $\alpha_p = \Sigma X_i \alpha_i$,

then $E(R_p) = \alpha_p + \beta_p E(R_m)$.

Consider portfolio P to be Market portfolio (M).

Then $E(R_p)$ must = $E(R_m)$.

For this to be true, $\alpha_p = 0$ and $\beta_p = 1$.

→ The Market's beta is 1.

If $\beta_i > 1$, stock is more risky than Market;
If $\beta_i < 1$, stock is less risky than Market.

β_i measures security's risk.

B. Portfolio risk, σ_p^2.

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

Observe:
In the double sum, $i \neq j$ yields cross-product terms.
If $i=j$, then the terms would be $X_i^2 \beta_i^2 \sigma_m^2$;
but these are just the terms in the first sum.

Thus, we can re-write as a single double-sum:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

Or $$\sigma_p^2 = \sum_{i=1}^{N} X_i \beta_i \sum_{j=1}^{N} X_j \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

Or $$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

C. The Security's contribution to σ_p^2.

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{\varepsilon i}^2$$

1. Consider naïve diversification, letting $X_i = 1/N$:

   Then $\sigma_p^2 = \beta_p^2 \sigma_m^2 + (1/N) \sum_{i=1}^{N} (1/N)\, \sigma_{\varepsilon i}^2$.

   The last sum is the avg residual (firm-specific) risk for all securities in the portfolio.

2. As $N \rightarrow \infty$, second term $\rightarrow 0$, and $\sigma_p^2 \rightarrow \beta_p^2 \sigma_m^2$.

   As N ↑, individual security risk is diversified away, but the risk from broad market movements remains.

3. Since σ_m^2 is the same, regardless of which stock, the factors that influence β_p are the relevant factors that contribute to portfolio risk.

4. Since $\beta_p = \Sigma X_i \beta_i$,
   the contribution of security i to σ_p is its β_i;

   ε_i and $\sigma_{\varepsilon i}^2$ are irrelevant to σ_p if we diversify.

5. Implications.

Again, the 'total risk' of an individual security is:

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_m^2}_{\substack{\uparrow \\ \text{nondiversifiable} \\ \text{systematic}}} + \underbrace{\sigma_{\varepsilon i}^2}_{\substack{\uparrow \\ \text{diversifiable} \\ \text{nonsystematic}}} .$$

Since diversifiable risk can be eliminated cheaply by simply ↑ing N, the Market will not pay a higher return for a stock with a higher $\sigma_{\varepsilon i}^2$.

What matters is the risk in a security that cannot be diversified away $\rightarrow$ ($\beta_i^2 \sigma_m^2$). Again, since σ_m^2 is the same for any security, i, β_i is the relevant measure of a security's risk, as well as the security's risk contribution to σ_p^2.

V. Estimating β_i.

To estimate $E(R_p)$ & σ_p^2 with Single-Index Model, we need forecasts of each firm's β_i, to estimate $E(R_i)$, σ_i^2, and the ρ_{ij} – the correlation matrix.

A. Estimating Historical β's.

The Single-Index Model: $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$

Observe $\{R_{mt}, R_{it}\}$ over time; estimate $\{\alpha_i, \beta_i, \& \sigma_{\varepsilon i}^2\}$.

1. Regression estimates of α_i and β_i :

$$\hat{\beta}_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\Sigma (R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m)}{\Sigma (R_{mt} - R_m)^2}$$

$$\hat{\alpha}_i = R_{it} - \beta_i \bar{R}_{mt}$$

2. Note: β_i is related to ρ_{im}.

$$\rho_{im} = \frac{\sigma_{im}}{\sigma_i \sigma_m} = \frac{\sigma_{im}}{\sigma_i \sigma_m} \frac{\sigma_m}{\sigma_m} = \frac{\sigma_{im}}{\sigma_m^2} \frac{\sigma_m}{\sigma_i} = \beta_i \frac{\sigma_m}{\sigma_i}$$

3. Consider $\sigma_{\varepsilon i}^2$.

Suppose $\hat{\alpha}_i = 4$ and $\hat{\beta}_i = .7$;
Then, if R_m↑ 1%, expect R_i to ↑ .7%.

Some obs. will lie above the line, some below.
If obs. close to estimated line, $\sigma_{\varepsilon i}^2$ small;
If obs. spread out around estimated line, $\sigma_{\varepsilon i}^2$ big.

Wire through pipe analogy; $\sigma_{\varepsilon i}^2$ = diameter of pipe.
More noise, less precision; less confidence in β_i.

$\sigma_{\varepsilon i}^2$ is the variance of deviations from line.

$$\text{Deviations} = [R_{it} - (\hat{\alpha}_i + \hat{\beta}_i R_{mt})] = e_{it}$$

$$\text{Var}(e_{it}) = (1/N) \sum_{t=1}^{N} [R_{it} - (\hat{\alpha}_i + \hat{\beta}_i R_{mt})]^2 = E(e_{it}^2).$$

B. Accuracy of Historical β's.

Reasonable to use historical data on R_i and R_m,
to estimate historical β_i as predictor of future β_i.

1. Problems:

   a. True β's nonstationary – change over time.
   Goal is to predict future β's – future ρ's!

   b. Regression Model estimates of β have 'error';
   – only estimates of true β.

2. Blume study:

   Computed historical β's over 2 different 7-year periods.

Number of Securities	Correlation
individual securities	.60
2-stock portfolios	.73
4-stock portfolios	.84
7-	.88
10-	.92
20-	.97
35-	.97
50-	.98

   Compared β's for each size portfolio across the 2 periods.
   How did β's change? What is correlation across periods?

   Found: 1) β's on large pfs changed little over time;
   2) β's on small pfs changed more over time.

   Thus, β's on small pfs contain less info about future β's.

3. Why Blume's result?

   a. Why might estimates of historical β change?

      i. because true β's change;

      ii. historical β estimates are measured with error.

   b. Confidence in historical β from regression model depends on how much noise, $\sigma_{\varepsilon i}^2$, around fitted line.

      i. The larger is $\sigma_{\varepsilon i}^2$:

         - the less confidence in historical β;

         - the less predictive power historical β will have for future β.

   c. Changes in historical β's for individual stocks will wash each other out in a portfolio.

      i. Changes in true β's will differ across stocks;

      ii. Errors in historical β estimates differ across stocks;

      iii. Some β estimates will ↑; some will ↓;

      iv. When combined into portfolio, individual changes & errors tend to cancel out.

      v. Thus should observe less change in portfolio β's.

   d. Portfolio β's are thus better predictors of future β's than are individual security β's.

C. Fundamental β's.

Firm's true risk reflects more than just how R_i responds to R_m (i.e., β_i).

1. Firm's fundamentals matter!

Firm's fundamentals should affect its risk (β):

(-) a. Dividend payout ratio (Div./Earnings);
(+) b. Asset growth;
(+) c. Leverage (Debt/Equity);
(-) d. Liquidity (Curr. Assets)/(Curr. Liab.);
(-) e. Asset Size;
(+) f. Earnings Variability;
(+) g. Earnings β (or Accounting β).

2. Cross-sectional Regression Model.

Consider info about firm's fundamentals to improve predictions of future β.

$$\beta_i = a_0 + a_1 X_{1i} + a_2 X_{2i} + \ldots + a_3 X_{Ni} + \varepsilon_i$$

where β_i = i^{th} firm's historical β;

X's represent N fundamental factors that may affect risk for i^{th} firm;

a_k = impact of k^{th} fundamental on β_i .

3. The fitted values = "fundamental β's".

4. Barr Rosenberg's Model.

   a. Combines information on historical β's
      with information on firm fundamentals.

   b. Among his 101 variables in regression Model:

      i. market characteristics of stock:
         1. previous estimates of firm's historical β;
         2. share trading volume;
         3. daily price trading range; …

      ii. measures of firm fundamentals:
         1. measures of Earnings variability;
         2. indicators of percieved success;
         3. firm size and age;
         4. measures of historical & perceived growth;
         5. measures of financial risk;
         6. other firm characteristics (listing, business…).

      iii. Industry dummy variables.
         To measure influence of industry characteristics
         on the firm's fundamental β.

   c. Difficult to interpret all the a_k estimates;
      Who cares? Goal is to forecast β.
      Seems to provide better forecasts than either
      historical β's or fundamental data alone.

5. Alternative Approach: Forecast X's, then β's.
   Forecast future firm fundamentals;
   Plug in forecasted X's to generate forecasts of β_i .

   a. With 101 variables, difficult task!
   b. Has been done with much simpler models.

6. Wells Fargo approach, E&G, Ch. 18.

D. The Market Model.

1. Same as Single-Index Model, except do not assume $E(\varepsilon_i \varepsilon_j) = 0$.

Basic Equation:	$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$;	$i = 1 - N$
By Construction:	$E(\varepsilon_i) = 0$;	$i = 1 - N$.
By Assumption:	$E[\varepsilon_i (R_m - \bar{R}_m)] = 0$;	$i = 1 - N$.
(NOT THIS)	$E(\varepsilon_i \varepsilon_j) = 0$;	for $i \neq j$.
By Definition:	$Var(\varepsilon_i) = \sigma_i^2$;	$i = 1 - N$.
	$Var(R_m) = E(R_m - \bar{R}_m)^2 = \sigma_m^2$.	

2. Estimating β_i is the same with the Single-Index Model or Market Model.

   a. Implications for portfolio risk are not same;

      Since Mkt Model does not assume $E(\varepsilon_i \varepsilon_j) = 0$, it does not lead to the simple expressions of portfolio risk implied by Single-Index Model. (See IV.B. discussion above.)

   b. Used in tests of Efficiency (E&G, Ch. 17).

VI. Summary

1. One goal is to forecast firm β's & judge individual firm risk.

2. Another goal is to forecast correlation matrix for N stocks.
   Recall: need forecasts of $E(R_i)$, σ_i^2 and ρ_{ij}
   to forecast $E(R_p)$ and σ_p^2 – opportunities locus.

3. Single Index Model simplifies process:
   - simplifies immense data problem;
   - resolves organizational problem (firms org. by industry).

4. How well does S-I Model forecast correlation matrix?
   S-I Model: [$\rho_{ij} = \sigma_{ij}/\sigma_i\sigma_j = (\beta_i\beta_j\sigma_m^2)/\sigma_i\sigma_j$].

   Studies compare forecasts from historical ρ's
   with forecasts from S-I Model using β's.

   Find β's forecast future ρ's better than historical ρ's.

   -- S-I Model outperforms historical correlations!

5. Single Index Model useful.
   a. Simpler than estimating entire correlation matrix;
   b. Forecasts future ρ's better than correlation matrix.