E&G, Ch. 5: Efficient Portfolios

Overview:
E&G, Ch. 10 developed the preference structure;
indifference curves in $[E(R_p), \sigma(R_p)]$ space.

E&G, Ch. 4 established relevant measures of portfolio return and risk.

E&G, Ch. 5 uses the results of Ch. 4 to develop the opportunity locus in a world where a portfolio's mean & variance are the objects of choice.

Then decision rules are developed to show how individuals choose optimal portfolios that max E(U) of Wealth.

I. The Opportunity Locus:
(1) No short selling;
(2) No lending or borrowing @ r_f.

A. Background.

1. For financial manager, operating risk of a firm can be measured by estimating the mean & variance of returns provided by the firm's portfolio of assets.

2. For portfolio manager, risk and return are the mean and variance of the weighted avg of assets in the portfolio.

3. To understand how to manage risk, must explore risk & return opportunities available by investing in different combinations of assets.

B. Consider portfolio with 2 assets.

Asset	Expected Return	Standard Deviation	Correlation
1	μ_1	σ_1	
2	μ_2	σ_2	ρ

1. **Portfolio**: w_1 in Asset 1; $w_2 = 1-w_1$ in Asset 2.

$$E(R_p) = E_p = w_1 \mu_1 + w_2 \mu_2$$

$$\sigma^2(R_p) = \sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$$

Or: $$\sigma_p^2 = w_1^2 \sigma_1^2 + (1-w_1)^2 \sigma_2^2 + 2 w_1 (1-w_1) \rho \sigma_1 \sigma_2$$

This expresses the risk of the portfolio in terms of the risks of the 2 assets (σ_1 & σ_2), their correlation (ρ), and w_1, how much to invest in Asset 1.

By changing w_1, we can change the risk of the portfolio.

2. **Minimum risk portfolio**,

Choose the weight, w_1, where $d\sigma_p^2/dw_1 = 0$:

$$d\sigma_p^2/dw_1 = 2w_1\sigma_1^2 + 2(1-w_1)(-1)\,\sigma_2^2 +$$
$$+ 2(1-w_1)\,\rho\,\sigma_1\,\sigma_2 + 2w_1(-1)\,\rho\,\sigma_1\,\sigma_2 = 0$$

Solve this for w_1^*:

(÷2) $0 = w_1\sigma_1^2 - (1-w_1)\sigma_2^2 + (1-w_1)\,\rho\,\sigma_1\,\sigma_2 - w_1\,\rho\,\sigma_1\,\sigma_2$

$$0 = w_1\sigma_1^2 - (1-w_1)\sigma_2^2 + (1-2w_1)\,\rho\,\sigma_1\,\sigma_2$$
$$0 = w_1\sigma_1^2 + w_1\sigma_2^2 - 2w_1\,\rho\,\sigma_1\,\sigma_2 - \sigma_2^2 + \rho\,\sigma_1\,\sigma_2$$
$$0 = w_1\,[\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\,\sigma_2] - [\sigma_2^2 - \rho\,\sigma_1\,\sigma_2]$$

$$w_1^* = [\,\sigma_2^2 - \rho\,\sigma_1\,\sigma_2\,] \,/\, [\,\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\,\sigma_2\,]$$

$$w_2^* = 1-w_1^* = [\,\sigma_1^2 - \rho\,\sigma_1\,\sigma_2\,] \,/\, [\,\sigma_1^2 + \sigma_2^2 - 2\rho\,\sigma_1\,\sigma_2\,]$$

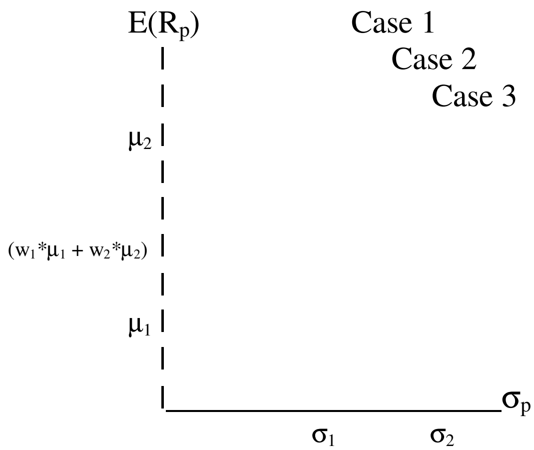
3. What does minimum risk portfolio mean?

This helps to reveal the **opportunity locus** in $[E_p\ \sigma_p]$ space; shows combinations of $[E_p, \sigma_p]$ available, given assets 1 & 2.

a. Entire pf in Asset 1,
   $w_1 = 1$, $w_2 = 0$,
   get (μ_1, σ_1).

b. Entire pf in Asset 2,
   $w_1 = 0$, $w_2 = 1$,
   get (μ_2, σ_2).

c. Use w_1^* and w_2^*,
   get min. risk poss.



If we diversify, can obtain lower risk.
How small we can make risk depends on ρ.

Three Cases help illustrate the Locus:

Case 1: $\rho = -1$; Opportunity Locus is 2 line segments.

Case 2: $\rho = 0$; Opportunity Locus is curve.

Case 3: $\rho = 1$; Opportunity Locus is 1 line segment.

Relevant Formulas to see Opportunity Locus:

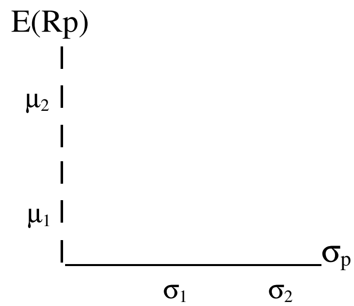
(1) $E_p = w_1 \mu_1 + w_2 \mu_2$

(2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$

(3) $w_1^* = [\sigma_2^2 - \rho\sigma_1\sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]$

(4) $w_2^* = [\sigma_1^2 - \rho\sigma_1\sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]$

<u>Case 1: ρ = -1.</u>



$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 - 2 w_1 w_2 \sigma_1 \sigma_2$
$= [w_1 \sigma_1 - w_2 \sigma_2]^2$

$\sigma_p = \pm [w_1 \sigma_1 - w_2 \sigma_2]$

Which root holds (+ or -) depends on which is > 0, to ensure $\sigma_p \geq 0$ as we vary w_1.

For $\rho = -1$, *portfolio risk is weighted difference of σ_1 & σ_2.*

As we vary w_1 from 0 to 1,
σ_p^2 varies along a parabola, but σ_p varies linearly.

E_p & σ_p are both linear, so **opportunities locus is linear**.

Actually, **opportunity locus is 2 line segments**;
for some values of w_1, segment has slope > 0;
for others, segment has slope < 0.

Consider the **minimum risk portfolio**:

$w_1^* = [\sigma_2^2 + \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2]$
$= \sigma_2 (\sigma_1+\sigma_2) / (\sigma_1+\sigma_2)^2 = \sigma_2 / (\sigma_1+\sigma_2)$

$w_2^* = [\sigma_1^2 + \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2]$
$= \sigma_1 / (\sigma_1+\sigma_2)$

Then, $\sigma_p = w_1^* \sigma_1 - w_2^* \sigma_2$
$= [\sigma_2 / (\sigma_1+\sigma_2)]\, \sigma_1 - [\sigma_1 / (\sigma_1+\sigma_2)]\, \sigma_2 = 0.$

Case 1: ρ = -1, continued.

Relevant Formulas to see Opportunity Locus:

(1) $E_p = w_1 \mu_1 + w_2 \mu_2$
(2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$

Minimum Risk Portfolio:

(3) $w_1^* = [\sigma_2^2 - \rho\sigma_1\sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]$
(4) $w_2^* = [\sigma_1^2 - \rho\sigma_1\sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]$

Recall the example from E&G, Chapter 4:

Table Dollars at Period 2 Given Alternative Investments

Condition of Market	Asset 2	Asset 3	Combination of [.6*Asset 2 + .4*Asset 3]
Good	$1.16	$1.01	$1.10
Average	$1.10	$1.10	$1.10
Poor	$1.04	$1.19	$1.10
$E(R_i)$	10%	10%	10%
$\sigma^2(R_i)$	24	54	0
σ_{ij}	-36		

We showed that this is an example of Case i ; That is, for Assets 2 & 3, ρ = -1.

But how did we know how to combine Assets 2 & 3? [i.e., $X_1 = .6$ and $X_2 = .4$]

Using the above formulas for the **minimum risk portfolio**:

(3) $w_1^* = [\sigma_2^2 - \rho\sigma_1\sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2]$
$= [54 - (-36)] / [24 + 54 - 2(-36)]$
$= [90] / [150]$
$= .6$

(4) $w_2^* = 1 - .6 = .4$

Relevant Formulas to see Opportunity Locus:

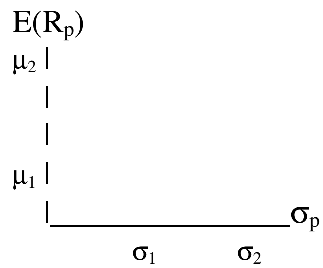
(1) $E_p = w_1 \mu_1 + w_2 \mu_2$

(2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$

(3) $w_1^* = [\sigma_2^2 - \rho \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2]$

(4) $w_2^* = [\sigma_1^2 - \rho \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2]$

Case 2: $\rho = 0$.



$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2$$

$$\sigma_p = [w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2]^{1/2}$$

As we vary w_1,
σ_p^2 varies along a parabola, and
σ_p varies according to the square root of a quadratic.
Here the **opportunity locus is not linear**.

Consider the **minimum risk portfolio**:

$$w_1^* = [\sigma_2^2] / [\sigma_1^2 + \sigma_2^2]$$

$$w_2^* = [\sigma_1^2] / [\sigma_1^2 + \sigma_2^2]$$

Then, (1) & (2) give $[E_P, \sigma_P]$ for the min. risk portfolio –
the location of the vertical point on the Opportunity Locus.

Relevant Formulas to see Opportunity Locus:

(1) $E_p = w_1 \mu_1 + w_2 \mu_2$

(2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$

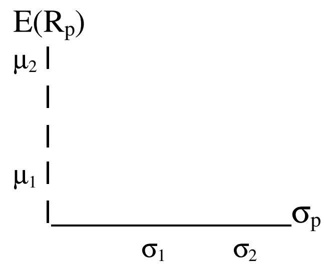
(3) $w_1^* = [\sigma_2^2 - \rho \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2]$

(4) $w_2^* = [\sigma_1^2 - \rho \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2]$

Case 3: $\rho = +1$.

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2$$
$$= [w_1 \sigma_1 + w_2 \sigma_2]^2$$

$$\sigma_p = \pm [w_1 \sigma_1 + w_2 \sigma_2]$$



For $\rho = +1$, *portfolio risk is weighted sum of* σ_1 *&* σ_2.
Again, as in Case 1, as we vary w_1 from 0 to 1,
σ_p^2 varies along a parabola, but σ_p varies linearly.

Because E_p & σ_p are both linear, **opp. locus is linear**.

Consider the **minimum risk portfolio**.

$$w_1^* = [\sigma_2^2 - \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2]$$
$$= \sigma_2 (\sigma_2 - \sigma_1) / (\sigma_2 - \sigma_1)^2 = \sigma_2 / (\sigma_2 - \sigma_1)$$

$$w_2^* = [\sigma_1^2 - \sigma_1 \sigma_2] / [\sigma_1^2 + \sigma_2^2 - 2 \sigma_1 \sigma_2]$$
$$= \sigma_1 / (\sigma_1 - \sigma_2)$$

NOTE: One of these denominators will be >0, & one <0.
Thus, one of these weights will be >1, & one <0.
Implies short selling is required to get min. risk portfolio.

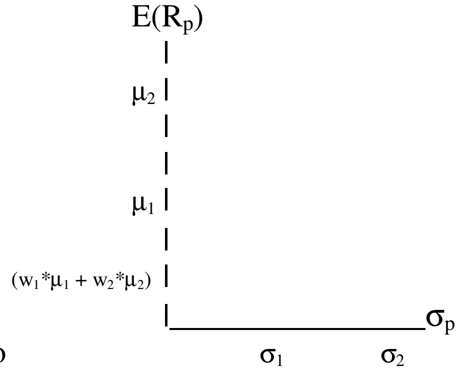
Then, $\sigma_p = w_1^* \sigma_1 + w_2^* \sigma_2$

$$= [\sigma_2 / (\sigma_2 - \sigma_1)] \sigma_1 + [\sigma_1 / (\sigma_1 - \sigma_2)] \sigma_2$$
$$= [\sigma_2 \sigma_1 / (\sigma_2 - \sigma_1)] + [\sigma_1 \sigma_2 / (\sigma_1 - \sigma_2)]$$
$$= [\sigma_2 \sigma_1 / (\sigma_2 - \sigma_1)] - [\sigma_1 \sigma_2 / (\sigma_2 - \sigma_1)] = 0.$$

4. Elaborate on Case 3.

a. Opp. Locus linear.

b. Min. risk portfolio:
   i. at w_1^* and w_2^*;
   ii. $w_1^*<0$, $w_2^*>1$ (or vice versa);
   iii. $\sigma_p = 0$.



Short selling allows us to extend the opportunities locus until it reaches vertical axis, where $\sigma_p = 0$.

c. In this case [***no short selling allowed***], the min. risk portfolio occurs when we put the entire portfolio in the asset with less risk.
   i. $w_1 = 1$ if $\sigma_1 < \sigma_2$;
   ii. $w_2 = 1$ if $\sigma_2 < \sigma_1$.

5. Note: Opportunity Locus is determined by the formulas for E_p and σ_p^2:

(1) $E_p = w_1 \mu_1 + w_2 \mu_2$

(2) $\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \rho \sigma_1 \sigma_2$

6. The minimum risk portfolio can easily be found:

(3) $w_1^* = [\,\sigma_2^2 - \rho \sigma_1 \sigma_2\,] / [\,\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2\,]$

(4) $w_2^* = [\,\sigma_1^2 - \rho \sigma_1 \sigma_2\,] / [\,\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2\,]$

7. Opportunity Locus lies between extreme cases ($\rho = -1$ and $\rho = +1$) and depends on ρ.

$E(R_p)$

$\rho = -1$

$\rho = +1$

σ_p

8. Put the Opportunity Locus with Indifference Curves, to get **solution to portfolio choice problem**.

a. What part of Locus is "efficient" for a risk averse investor?

b. Indifference Curves that cross the Locus in the portion, AC are dominated by a higher Curve that crosses the Locus in the portion, AB.

c. AB is **Efficient Frontier**.

$E(R_p)$

σ_p

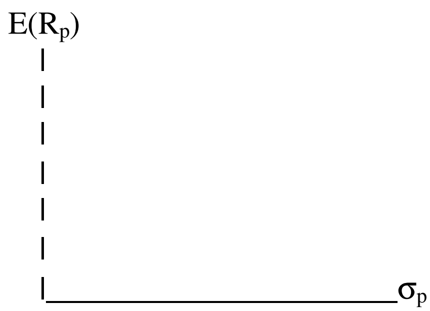
C. Consider the assumptions.

1. No short selling ($w_i \geq 0$).
2. No lending or borrowing at r_f.

We are also assuming investors:

3. have homogeneous beliefs about $E(R_i)$, σ_i, and ρ_{ij}, so that everyone faces the same Opportunity Locus.
4. have different preference structures.

For example:
I_1 more risk-averse.
demands much higher E_p
for a little more σ_p.
I_2 less risk averse.
accepts much higher σ_p
for a little more E_p.



Later, when we relax assumption (2), we'll find all investors choose the same optimal portfolio, even though they have different preferences toward risk. (Portfolio Separation)

D. In practice, Opportunity Locus can be found by solving either of the following math programming problems:

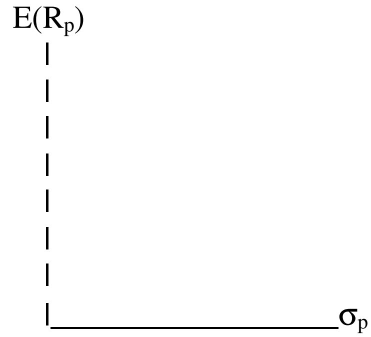
I. MIN σ_p^2 subject to $E(R_p) = K$;
II. MAX $E(R_p)$ subject to $\sigma_p^2 = K$.

Note: I. gives Minimum Variance Opportunity Locus;
II. gives the Efficient Frontier.

E. More than 2 Risky Assets.

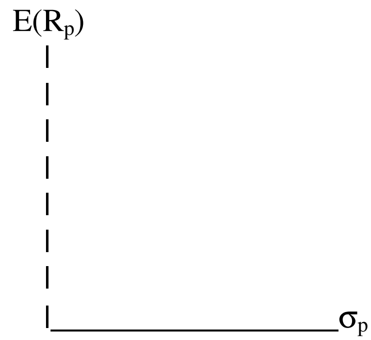
1. With 3 Risky Assets, There are 3 possible Frontiers for any 2-asset portfolios.

2. Consider combinations of all 3 assets, there are an infinite number of possible frontiers in this moon-shaped area, for different weights on the three assets.



3. Again, for risk-averse, Efficient Frontier is top edge, from vertical point (min. risk pf) to point C.

4. Analysis can be extended to N risky assets; implies similar Efficient Frontier.



Major Point: Investor maximizes utility by diversifying along the Efficient Frontier.

Note: With N assets, need lots of information & computations. Will simplify this later.

II. Opportunity Locus with Short Sales Allowed.

A. Definition of Short Selling:

Selling a security that you do not own;
Taking a 'negative' position in a security ($W_i < 0$).

B. Example:

Stock ABC selling at $100/share, pays $3 dividend.
You expect it to go down to $95. Alternatives:

	Now	Later		Now	Later
Buy:	-100		Sell:	+100	
Receive Div.:		+3	Pay Div:		-3
Sell:	____	+95	Buy:	____	-95
	-100	+98		+100	-98

Don't buy now! Now - borrow share & sell;
Want negative amt! Later - pay div., buy back, return.

Note: lender of share is no worse off;
borrower has created a security that has the opposite characteristics of buying a share.

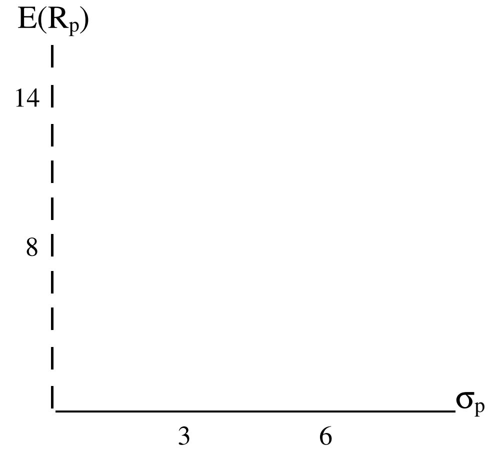
[Actually, lender (brokerage firm) charges interest for lending stock, and demands collateral.]

Note: Shorting makes sense if you expect price to drop.
If expectations are wrong – price increases,
Must buy back later at higher price – lose.

C. Short selling may also make sense if $E(R_p) > 0$.

1. Consider 2 assets:
   $E_A = 8\%$; $\sigma_A = 3$;
   $E_B = 14\%$; $\sigma_B = 6$;
   $\rho = 0.5$.

   w_1 = part of Wealth in A; $(1-w_1)$ in B.



2. **With no short sales**, $0 \leq w_1 \leq 1$, and Opportunity Locus goes from A to B.

3. **With short sales allowed**,
   can put all of initial wealth in B,
   and can sell short A and put the additional cash flow into B.

   Simply allow w_1 to be < 0 or > 1.
   In this case, $w_1 < 0$ and $w_2 > 1$.

   This will increase E_p, but it will also increase σ_p.

   Point: Opportunity Locus extends beyond points A and B
   if short sales are allowed.

4. Whether investor will take a short position depends on his preference structure.

Efficient Frontier now ranges from vertical (min. σ_p) point upward.

Investor I_1
more risk averse,
steeper curves.
Will buy some A
and some B.

Investor I_2
less risk averse,
flatter curves.
Will sell A short & invest more than initial wealth in B.

$E(R_p)$

σ_p

III. Opportunity Locus with borrowing or lending at r_f.

A. Graphical Explanation.

1. Can borrow or lend at riskfree rate, r_f. [Can get zero risk by putting entire wealth riskfree asset, Pt A.]

2. Can invest in any combination of risky assets in moon-shaped area;

E(R_p)

σ_p

Can combine 1) & 2); can get any point on a ray extending from Pt A thru any point in moon-shaped area.

Risk averse investor will rotate this ray upward and left, until ray is tangent to moon-shaped area at Point P.

Choose portfolio P as 'optimal' portfolio of risky assets; then either lend or borrow at the riskfree rate to max E(U).

Lending at r_f = buying T. Bills;
Borrowing at r_f = shorting T. Bills.

Investor 1 is more risk averse;
Will put some in portfolio P and some in riskfree asset (at r_f).

Investor 2 is less risk averse;
Will put all of initial wealth in portfolio P,
and will then borrow more at r_f to invest more in portfolio P.

B. Mathematical Explanation.

1. Let w_1 = portion of initial wealth in Portfolio P.

   $(1-w_1)$ = portion in riskfree asset.

   Call this combination Portfolio Z.

   [$w_1 > 1$ means that investor borrows at r_f,
   and invests more than initial wealth in Portfolio P.]

2. The expected return of this combination:

   $E(R_Z) = (1-w_1)\, r_f + w_1\, E(R_P)$ [linear in $E(R_P)$]

3. The risk of this combination:

$$\sigma^2(R_Z) = (1-w_1)^2\, \sigma_f^2 + w_1^2\, \sigma_P^2 + 2\,(1-w_1)w_1\, \rho_{fP}\sigma_r\, \sigma_P$$

$$= w_1^2\, \sigma_P^2 \qquad [\sigma_f = \rho_{fP} = 0]$$

or: $\sigma_Z = w_1\, \sigma_P$ [linear in σ_P]

4. Combining $\sigma^2(R_Z)$ and $E(R_Z)$ yields Opportunity Locus:

   Solving for w_1: $w_1 = \sigma_Z / \sigma_P$;

   Substituting into $E(R_Z)$:

$$E(R_Z) = (1 - \sigma_Z/\sigma_P)\, r_f + \sigma_Z/\sigma_P\, E(R_P)$$

or: $$E(R_Z) = r_f + [\{E(R_P) - r_f\} / \sigma_P]\, \sigma_Z$$

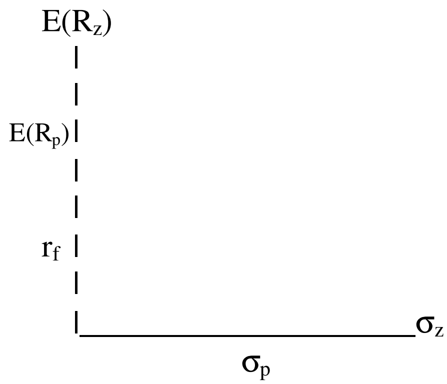
Equation of Ray -- Efficient Frontier.

C. Capital Market Line.

Efficient Frontier: linear relation between risk and return.

$$E(R_Z) = r_f + [\{E(R_P) - r_f\} / \sigma_P]\, \sigma_Z$$

Observe:



1. Intercept = r_f.
   If w_1=0, (1-w_1)=1,
   put all in T. Bills; $\sigma_P = 0$.

2. Slope = $[\{E(R_P) - r_f\}/\sigma_P]$
   Reflects a "price of risk";
   Shows how much $E(R_Z)$
   must increase along
   Efficient Frontier, if σ_Z increases one unit.

   Intuition: $\{E(R_P) – r_f\}$ is "risk premium" of P over r_f.
   To accept more risk, σ_P, need compensation, $\{E(R_P) – r_f\}$.

3. Ray passes through $[E(R_P), \sigma_P]$;
   If $w_1 = 1$, invest all of initial wealth in P.
   Above P, borrow at r_f; Below P, lend at r_f.

D. Portfolio P is special.

1. If all face the same opportunity locus & riskfree rate, r_f,
   Then all will rotate ray from r_f to same tangency, at P.
   a. All investors will hold the same pf of risky assets, P.
   b. P is the solution to the portfolio selection problem.
   c. Choice of P is independent of individual preferences.
      All will invest in P and then either borrow or lend,
      depending on their preferences.

2. P is the **Market Portfolio**.
   a. All assets are held in P according to their market values.
   b. Weights in portfolio P: $w_i = V_i / \Sigma V_i$,
      where V_i = mkt value of ith asset,
      and ΣV_i = total mkt value of all assets.

3. **Mkt Equilibrium** is reached when P is Mkt portfolio;
   asset prices must align so that there is no excess demand for any asset; r_f must equate agg. borrowing and lending.

4. **Separation Theorem** – the ability to determine optimal pf
   without knowing anything about individual preferences.
   a. The decision of what combination of risky assets to buy
      is separated from the decision whether to borrow or lend.
   b. Result rests on assumptions:
      i. all face same opportunity locus of risky assets;
      ii. all can borrow or lend at same riskfree rate, r_f.

5. If you can lend at r_f, but **cannot borrow at r_f**,
   Separation Theorem breaks down,
   and efficient frontier becomes APC below.

6. If you can lend at r_f, & borrow at higher rate, r_B,
   then efficient frontier becomes the following.

E. The benefit of capital markets.

1. Can think of a "Capital Market" as simply the opportunity to borrow or lend at r_f, as in E&G, Ch. 1, combined with the opportunity to invest in risky assets.

2. Without a riskfree asset, there is no opportunity to exchange riskless investing for risky investing; i.e., no opportunity to borrow or lend.

   a. then individuals are bound by the efficient frontier representing opportunity locus of all risky assets.

3. Adding a riskfree asset means individuals can attain opportunities beyond common convex opportunity locus, by investing in P and borrowing or lending.

Given opportunity locus, E(R_p)
and preference structure, |
and initial endowmt at A; |
|
Initially move along |
efficient frontier toward B. |
|
w/o riskless asset, stop @ B. |
|
With riskless asset, |________________________ σ_p
can do better;

Continue on eff. frontier to Portfolio P, then borrow to reach C.

F. Three Important Results.

1. Almost everyone is better off with capital markets;
   And no one is worse off (Pareto Optimal).

2. Separation obtains – simplifies selection of optimal pf.

3. MRS between $E(R_P)$ and σ_P = Market Price of Risk,

   a. $MRS_i = MRS_j = \{E(R_P) - r_f\} / \sigma_P = MRT$;

   b. Same for all, regardless of attitude toward risk.

G. The Problem of Measuring Risk.

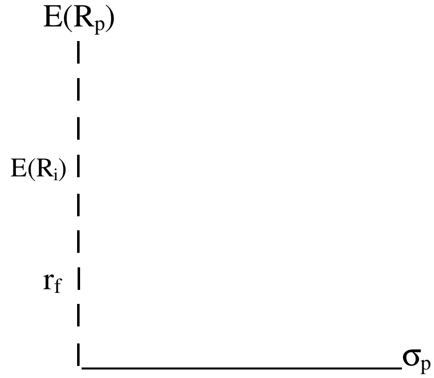
1. This discussion establishes that σ_P is the
   appropriate measure of risk for efficient portfolios.

2. However, σ_i is NOT the appropriate measure
   of risk for individual assets or inefficient portfolios.

3. Also, it is inappropriate to *compare*
   the σ_i of an individual asset with
   the σ_P of an efficient portfolio.

4. It's necessary to distinguish between portfolio risk (σ_P)
   and the contribution of a single asset to the riskiness of
   a well-diversified portfolio (i.e., the β of a security).

5. Graphically,
Consider portfolios I, J, & K,
e.g., individual securities.
(inefficient – inside CML).



a. Same $E(R_i)$, different σ_i.

b. If σ_i were an appropriate measure of risk for individual securities or inefficient portfolios, pf I should have higher $E(R_I)$ due to higher σ_I.

It doesn't! This violates the Law of One price.
(All securities or pf's that have same *joint distributions* of return must have same price in equilibrium).

I, J, & K are not on CML – inefficient!
Thus, σ_I is not their appropriate measure of risk.
Must consider *joint distribution* with all other assets!
Need variances and covariances of all assets!

σ_I includes *diversifiable risk* that market will not compensate, as well as *nondiversifiable risk* that market will compensate.

6. Consider $Var(R_P)$, with naïve diversification ($w_i = 1/N$).

$$\sigma^2_P = \Sigma\, w_i^2 \sigma_i^2 + \Sigma\Sigma\, w_i w_j \sigma_{ij}$$

$$= (1/N)\,[\bar{\sigma}_i^2] + ((N-1)/N)\,[\bar{\sigma}_{ij}]$$

As N increases, $\sigma_P^2 \rightarrow \bar{\sigma}_{ij}$.

7. Another way to examine the risk of a single asset, i, is to consider its contribution to pf risk, σ_P^2.

$$[d\sigma_P^2] / [dw_i] = 2\, w_i \sigma_i^2 + 2 \sum_{j=1}^{N} w_j \sigma_{ij}$$

Again, let $w_i = 1/N$; As $N \rightarrow \infty$, $w_i \rightarrow 0$, and $\Sigma\, w_i = 1$.

Thus, first term disappears, and second term remains, which is covariance of i^{th} asset with other assets in pf.

a. For well-diversified pf, the appropriate measure of the contribution of a single asset to market risk is its covariance with the market pf.

b. Even though individual investors may not hold well-diversified pfs, this is still appropriate measure.

c. The mkt does not compensate diversifiable risk.