E&G, Ch. 4: The Investment Opportunity Set under Risk

Overview.

Recall the problem.
Investor will buy pf that max. E(U) s.t. available opportunities.
Half the problem – *preference structure* for risk averse investor.
Next, must construct *opportunities locus*, in [E(R), σ(R)] space.

I. Measures of expected return, E(R), & risk, σ(R).

Any asset has several possible outcomes (returns),
Depending on what state of the world occurs.

The possible returns & their probabilities are summarized
in the probability distribution function (pdf) of asset return.

A. Properties of Expected Returns, E(R):

1. $E(R_{1i} + R_{2i}) = E(R_{1i}) + E(R_{2i})$.
2. $E(c*R_{1i}) = c*E(R_{1i})$.

Table 2. Return on Various Assets

Event	Probability	Asset 1	Asset 2	Asset 3
A	1/3	14	28	42
B	1/3	10	20	30
C	1/3	6	12	18
Expected Return		10	20	30

a. Asset 3 = Asset 1 + Asset 2; property 1 holds.

b. Asset 3 = 3*Asset 1; property 2 holds.

B. Properties of Risk, $\sigma^2(R)$.

Example, Table 3. Shows possible returns on 5 assets.

1. **Observe, for Asset 1**:

$E(R_{1i}) = 1/3(15 + 9 + 3) = 9$

$$\sigma^2(R_{1i}) = 1/3(15-9)^2 + 1/3(9-9)^2 + 1/3(3-9)^2$$
$$= 1/3(36) + 1/3(0) + 1/3(36)$$
$$= 24$$

2. **Compare Assets**:

a. Assets 1 & 2: $E(R_1) < E(R_2)$; $\sigma^2(R_1) = \sigma^2(R_2)$.

Choose Asset 2!
Indifference Curves = $f[E(R), \sigma(R)]$.
If $E(R_2)$ is higher and risk is the same, …

b. Assets 2 & 3: $E(R_2) = E(R_3)$; $\sigma^2(R_2) < \sigma^2(R_3)$.

Choose Asset 2!
Indifference Curves = $f[E(R), \sigma(R)]$.
If $E(R_2)$ is same and risk is less, …

c. Note: Asset 3 is different:

1. higher risk [$\sigma^2 = 54$];
2. when mkt is 'good', does poorly;
3. when mkt is 'bad', does well.

3. **Consider Portfolio Choices**:

Comparison of E(R) and σ(R) is easy, with just 2 assets.
More complicated when more possibilities considered.

The Variance (Risk) of a Combination of Assets

Recall, Expected Value of a sum = sum of Expected Values.

However, Variance of a sum ≠ sum of Variances;

Example 1: Consider Assets 2 & 3 from Table 3.
Suppose you have $1 to invest;
Can either buy Asset 2 or 3, or some combination.
See Table 4.

For either asset 2 or 3, the possibilities vary.
For the portfolio, [(.6)Asset 2 + (.4)Asset 3],
you are assured a return of 10% with no risk.

If Mkt good, Asset 2 does well, Asset 3 does poorly;
If Mkt avg, Assets 2 & 3 both do 10%;
If Mkt bad, Asset 3 does well, Asset 2 does poorly.

Asset 3 is hedge against poor Mkt performance.
This portfolio has zero risk because
Assets 2 & 3 are ***perfectly negatively correlated***.

Point1: The distribution of outcomes, [E(R), σ(R)], on portfolios of assets can be different from the distributions on the individual assets.

Point2: Which asset would you choose?

.	Asset 2	Asset 3	[(.6)Asset 2 + (.4)Asset 3]
$E(R_p)$	10%	10%	10%
$\sigma^2(R_p)$	24	54	0

Example 2: Consider a combination of Assets 2 & 4.

Identical distributions? Same E(R) and σ(R), but:

- Asset 2 depends on Mkt performance;
- Asset 4 depends on Rainfall.

Assume: Mkt performance & Rainfall independent.

- if Rainfall plentiful, Mkt may be good, avg, or bad.
- likewise if Rainfall is avg or poor.

As a result, returns on Assets 2 & 4 are ***independent***.

Consider the portfolio, [(.5)Asset 2 + (.5)Asset 4].

Same E(R), but lower risk than either Asset 2 or 4.

Intuition:

With either Asset 2 or 4,
extreme outcomes are .16 and .04, each with $p_i = 1/3$.

With portfolio, [(.5)Asset 2 + (.5)Asset 4],
extreme outcomes still .16 & .4, but now with $p_i = 1/9$.

- more possibilities closer to the mean;
- less dispersion about the mean;
- less risk (smaller σ(R)).

With Example 1, good outcomes of Asset 2 coincided with
bad outcomes of Asset 3 (***perfect negative correlation***),
so combination has 0 - risk.

With Example 2, good outcomes of Asset 2 ***independent*** of
good or bad outcomes of Asset 4,
so combination has less risk than 2 or 4 (but not 0).

Example 3: Consider a combination of Assets 2 & 5.

Identical distributions! Same E(R) and σ(R), but:

- Now both depend on Mkt performance.
- ***Perfect positive correlation***.

Result: Portfolio will yield same return as 2 or 5;

$E(R_p) = 10\%$; $\sigma^2(R_p) = 24$.

The characteristics of the portfolio are same as those of the individual assets;
Holding this portfolio does not change risk.

Summary:

Examples 1-3 illustrate the extreme possibilities.

If correlation = +1, portfolio risk is not reduced at all.

If correlation = between 1 & 0, risk is reduced some;

If correlation = between 0 & -1, risk is reduced more;

If correlation = –1, risk can be reduced to 0.

II. Characteristics of a Portfolio.

A. Measures of Portfolio Return.

1. Define Portfolio: splitting Wealth across N assets:
   a. X_i = weight (fraction of W) applied to i^{th} asset.
   b. $\Sigma X_i = 1$.

2. Actual Realized Return on Portfolio (ex post):
   a. $R_p = \Sigma X_i R_i$
      where R_i = actual return on i^{th} asset.

3. Expected Return on Portfolio (ex ante):
   a. $E(R_p) = E(\Sigma X_i R_i)$
      $= \Sigma E(X_i R_i)$ by Property 1
      $= \Sigma X_i E(R_i)$ by Property 2

First moment is 'nice' because of Properties 1 & 2;

Expected value of linear combination of random variables
= linear combination of expected values.

Second moment is more complex;
variance of l.c. of random variables $\neq$ l.c. of variances.

B. Measures of Portfolio Risk.

1. First consider l.c. of 2 assets.

$$\sigma_p^2 = \sigma^2(R_p) = E[R_p - E(R_p)]^2$$

$$= E[(X_1R_1 + X_2R_2) - (X_1 E(R_1) + X_2 E(R_2))]^2$$

$$= E[(X_1(R1 - E(R1)) + X_2 (R_2 - E(R_2))]^2$$

$$= E[X_1^2 (R_1 - E(R_1))^2 + X_2^2 (R_2 - E(R_2))^2 + \\ + 2X_1X_2 (R_1 - E(R_1))(R_2 - E(R_2))]$$

$$= X_1^2 E(R_1 - E(R_1))^2 + X_2^2 E(R_2 - E(R_2))^2 + \\ + 2X_1X_2 E(R_1 - E(R_1))(R_2 - E(R_2))$$

$$\boxed{\sigma_p^2 = \sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}}$$

a. Recall: σ_{12} = covariance;

if R_1 & R_2 positively related, $\sigma_{12} > 0$;
if R_1 & R_2 negatively related, $\sigma_{12} < 0$;
if R_1 & R_2 independent, then $\sigma_{12} = 0$.

b. Important!

$\sigma^2(R_p)$ can be less than σ_1^2 or σ_2^2 because:
i. σ_1^2 is multiplied by a fraction, X_1^2;
ii. σ_2^2 is multiplied by a fraction, X_2^2;
iii. σ_{12} is multiplied by 2 fractions, X_1 & X_2;
iv. σ_{12} may be small or negative.

c. Also recall: correlation = $\rho_{12} = \sigma_{12} / \sigma_1\sigma_2$; So $\sigma_{12} = \rho_{12}\,\sigma_1\sigma_2$;

Thus, $$\boxed{\sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\,\rho_{12}\,\sigma_1\sigma_2}$$

$$\sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\,\rho_{12}\,\sigma_1\sigma_2$$

1.d. Example:

Table	Dollars at Period 2 Given Alternative Investments		
Condition of Market	Asset 2	Asset 3	Combination of [.6*Asset 2 + .4*Asset 3]
Good	$1.16	$1.01	$1.10
Average	$1.10	$1.10	$1.10
Poor	$1.04	$1.19	$1.10

Case i: $\rho = -1$ [Assets 2 & 3; let $X_1 = .6$ and $X_2 = .4$].

We already showed that this portfolio gets $E(R_p) = 10\%$ with $\sigma^2(R_p) = 0$.

$$\sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\,\rho_{12}\,\sigma_1\sigma_2$$

$$= (.6)^2\,(24) + (.4)^2\,(54) + 2(.6)(.4)\,(-1)\,\{(24)(54)\}^{1/2} = 0.$$

This verifies the Table above; this portfolio gets $E(R_p) = 10\%$ with **0 risk**.

Case ii: $\rho = 0$ [Assets 2 & 4; let $X_1 = .5$ and $X_2 = .5$].

$$\sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\,\rho_{12}\,\sigma_1\sigma_2$$

$$= (.5)^2\,(24) + (.5)^2\,(24) + 2(.5)(.5)\,(0)$$

$$= ¼ * 24 + ¼ * 24 = 12.$$

This verifies our earlier result; this portfolio gets $E(R_p) = 10\%$ with **½ the risk**.

Case iii: $\rho = +1$ [Assets 2 & 5; let $X_1 = .5$ and $X_2 = .5$].

$$\sigma^2(R_p) = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\,\rho_{12}\,\sigma_1\sigma_2$$

$$= (.5)^2\,(24) + (.5)^2\,(24) + 2(.5)(.5)\,(+1)\,\{(24)(24)\}^{1/2}$$

$$= ¼ * 24 + ¼ * 24 + 2\,(¼)\,(24) = 24.$$

This verifies our earlier result; this portfolio gets $E(R_p) = 10\%$ with the **same risk**.

2. Next consider portfolio of 3 assets.

$$\sigma_p^2 = E[R_p - E(R_p)]^2$$
$$= E[\{X_1R_1+X_2R_2+X_3R_3\} - \{X_1E(R_1)+X_2E(R_2)+X_3E(R_3)\}]^2$$
$$= E[X_1(R_1-E(R_1)) + X_2(R_2-E(R_2)) + X_3(R_3-E(R_3))]^2$$
$$= E[X_2^2(R_1-E(R_1))^2 + X_2^2(R_2-E(R_2))^2 + X_3^2(R_3-E(R_3))^2] +$$
$$+ 2X_1X_2(R_1-E(R_1))(R_2-E(R_2))$$
$$+ 2X_1X_3(R_1-E(R_1))(R_3-E(R_3))$$
$$+ 2X_2X_3(R_2-E(R_2))(R_3-E(R_3))\ ]$$

or $\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 +$

$$+ 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_2X_3\sigma_{23}$$

3. Generalizing to a portfolio of N assets:

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_iX_j\sigma_{ij}$$

C. Risk-Reduction Benefits of Diversification.

1. The risk averse investor should diversify.

   a. Hold 1 stock – risky: $\sigma_p^2 = \sigma_1^2$.

   b. Hold 2 stocks – less: $\sigma_p^2 = X_1^2\,\sigma_1^2 + X_2^2\,\sigma_2^2 + 2X_1X_2\,\sigma_{12}$

   c. Hold N stocks – less: $\sigma_p^2 = \Sigma\ X_i^2\,\sigma_i^2 + \Sigma\,\Sigma\ X_iX_j\,\sigma_{ij}$

2. Extent of risk reduction possible depends on avg correlation, σ_{ij}.

3. Consider naïve diversification: let $X_i = 1/N$.

$$\sigma_p^2 = \sum_{i=1}^{N} (1/N)^2\,\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} (1/N)^2\,\sigma_{ij} \quad *(N-1)/(N-1)$$

$$= (1/N) \sum_{i=1}^{N} \sigma_i^2/N + (N-1)/N \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} \sigma_{ij} / N(N-1)$$

$$\sigma_p^2 = (1/N)\,[\bar{\sigma}_i^2] + (N-1)/N\,[\bar{\sigma}_{ij}]$$

where $\bar{\sigma}_i^2$ = average variance across all N assets;

$\bar{\sigma}_{ij}$ = average covariance between any 2 assets.

$$\sigma_p^2 = (1/N)\,[\bar{\sigma}_i^2] + (N-1)/N\,[\bar{\sigma}_{ij}]$$

(nonsystematic) (systematic, β)

4. As we diversify (N increases), $\sigma_p^2 \rightarrow \bar{\sigma}_{ij}$.

   Individual stock's $\bar{\sigma}_i^2$ can be diversified away,
   But ***covariance risk cannot be avoided***.

5. Suppose all N assets are independent;

   i.e., $\sigma_{ij} = 0$ for all i and j; so that $\bar{\sigma}_{ij} = 0$.

   Here, $\sigma_p^2 = (1/N)\,\bar{\sigma}_i^2$.

   By simply adding more (independent) assets,
   Can reduce risk toward zero.

   a. Problem: difficult to find many independent assets.
      Many common factors affect all assets.
      There is generally positive correlation across assets.

6. See Graphs.
   U.S., U.K., all international markets.