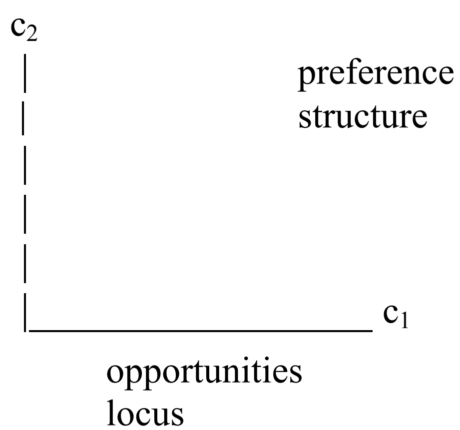
E&G, Chap 10 - Utility Analysis; the Preference Structure, Uncertainty
- Developing Indifference Curves in {E(R),σ(R)} Space.

A. Overview.

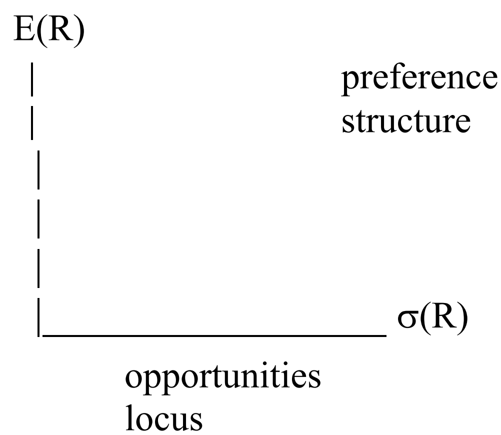
1. With Certainty, objects of choice – (c_1, c_2)

2. With Uncertainty, objects of choice – {E(R), σ(R)}

Certainty



Uncertainty



I. E&G, Chap 10 – Preference Structure in {E(R), σ(R)} space.

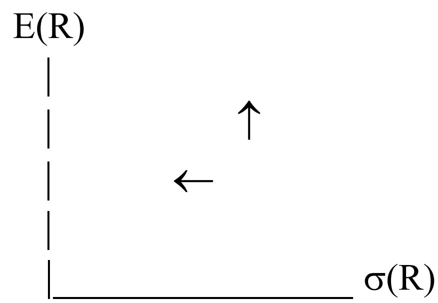
II. Then E&G, Chap 4, 5 – Opportunities Locus in {E(R), σ(R)} space.

III. Put I. & II. Together.
   A. Solution.
   B. Equilibrium.
   C. Relation between E(R) & σ(R).
   D. CAPM.

B. Indifference Curves in {E(R), σ(R)} space.

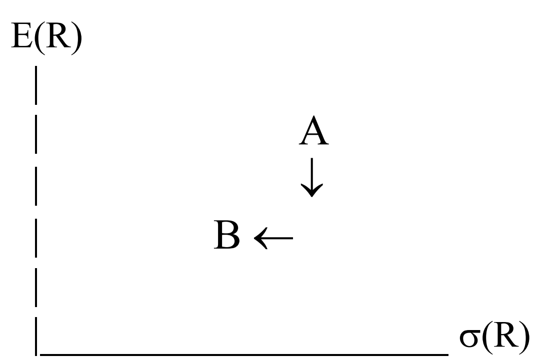
1. Assumptions:

    a. Happier if E(R) ↑.
    b. Happier if σ(R) ↓.
        i. Risk Averse.
    c. Dimishing MU.



E&G, Chap 10 discusses meaning of 1.a.-c.

2. Implications – indifference curves.
    a. Sope upward to right.
        i. A to B: lower E(R); must have lower σ(R) to remain indifferent.
    b. Concave upward.
        i. Give up equal increments of E(R);
        Must have increasing increments of reduced σ(R).



E&G, Chap 10 - Utility Analysis: Uncertainty
- Developing Indifference Curves in $\{E(R),\sigma(R)\}$ Space.

I. Specifying Preference Functions.

A. Simple Example.

1. Evaluating 2 risky assets, A & B:

Investment A		Investment B	
(W_i)	P_i	(W_i)	P_i
15	1/3	20	1/3
10	1/3	12	1/3
5	1/3	4	1/3

Define:
Expected Value of A = $E(A) = \Sigma P_i*W_i$
Variance of A = $\sigma^2(A) = \Sigma P_i*(W_i - E(A))^2$

$E(A) = 10$;
$\sigma^2(A) = 1/3(5)^2 + 1/3(0)^2 + 1/3(-5)^2 = 16.7$; $\sigma(A) = 4.08$

$E(B) = 12$;
$\sigma^2(B) = 1/3(8)^2 + 1/3(0)^2 + 1/3(-8)^2 = 42.7$; $\sigma(A) = 6.53$

Observe: $E(A) < E(B)$ -- bad for A
$\sigma(A) < \sigma(B)$ -- good for A

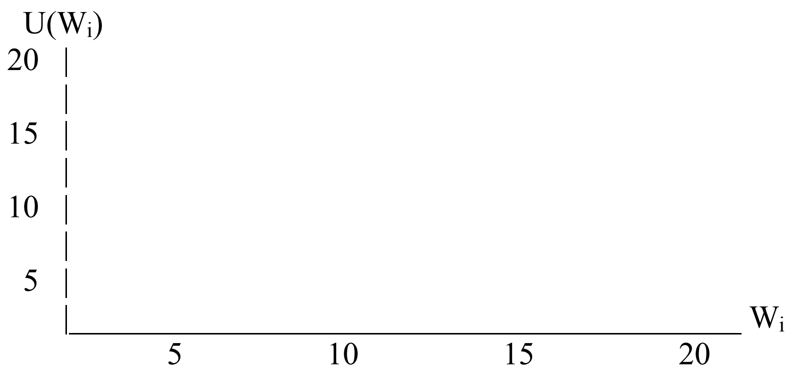
Risky assets!
We will assume investor is risk-averse;
demands higher expected value for asset with higher σ.

2. How to choose between Assets A & B?
   Wish to quantify this choice;
   consider how much you value [E(A), σ(A)] vs [E(B), σ(B)].
   Utility function does this by weighing possible outcomes.

   e.g., Consider following utility function – [$U(W_i)$]:
   Quantifies how much each value of W_i is worth to investor.

(1) (Wi) Outcome	(2) Utility Fn Weight(i)	(3) Value of Outcome
20	0.9	18
15	1.0	15
12	1.1	13.2
10	1.2	12
5	1.4	7
4	1.5	6

→ $U(W_i)$ = (3) = (1)*(2)



What should $U(W_i)$ depend on? $E(W_i)$ and $\sigma(R_i)$.

Observe: $U'(W) > 0$; More is better;
$U''(W) < 0$; Diminishing MU(W).

3. Given the utility function, how can we quantify the *value* or *utility* of the *expected* outcomes of A vs B?

   i.e., how can we rank A & B?

Define: Expected Utility = E(U) = $\Sigma\ P_i{*}\ U(W_i)$.

E(U) of A = 1/3 U(15) + 1/3 U(10) + 1/3 U(5)
= 1/3(15) + 1/3(12) + 1/3(7)
= 34/3

E(U) of B = 1/3 U(20) + 1/3 U(12) + 1/3 U(4)
= 1/3(18) + 1/3(13.2) + 1/3(6)
= 37.2/3

Makes sense –
If W_i = 10 occurs, U(10) = 12;
To get Expected Utility, weight $U(W_i)$ by P_i and sum.

Here investor would choose B, since it offers higher expected utility.

Will assume that investors choose among alternatives by maximizing E(U).

B. Specifying the functional form of $U(W_i)$.

1. Consider a quadratic utility function:

   $U(W) = aW - bW^2$

2. Example; Let a = 4, b = .1;

   Then: $U(W) = 4W - .1W^2$

   For any possible outcome, Wi,
   this function quantifies its utility.

3. Given this utility function,
   consider 3 risky investments with
   different expected returns & variances;
   Choose the one that maximizes E(U).

II. Economic Properties of Utility Functions.

A. Again, consider quadratic utility function:

- $U(W) = aW - bW^2$
- $U'(W) = MU(W) = a - 2bW$
- $U''(W) = -2b$

First economic property:

1. *Marginal Utility of Wealth* [U'(W) = MU(W)].

   a. If $U'(W) > 0$, $MU(W) > 0$; more is better.

   b. Therefore require $(a-2bW) > 0$; or $a > 2bW$.

Second economic property:

2. *Attitude Toward Risk* [U''(W)].

   a. Three possibilities:
      i. risk averse [$U''(W) < 0$];
      ii. risk neutral [$U''(W) = 0$];
      iii. risk loving [$U''(W) > 0$].

   b. With quadratic utility fn, this depends on b:
      i. if $b > 0$; concave down;
      ii. if $b = 0$; linear;
      iii. if $b < 0$; concave up.

   c. Case i. - diminishing MU(W);
      - $U''(W) < 0$;
      - require $b > 0$.

B. Digression: for intuition, define *Fair Gamble*.

1. Consider two choices:

A. Invest		B. Do Not Invest	
Outcome	Probability	Outcome	Probability
\$2	½	\$1	1
\$0	½		

2. A is lottery ticket;
   Write as (W1, W2; p) = (\$2, \$0; ½).

   B is riskless asset; (\$1; 1).

3. Assume: must pay \$1 for A, or keep the \$1 (B).

4. Observe: E(A) = E(B) = \$1.

   If you invest in A, may win or lose,
   But expected outcome equals the cost, \$1.

5. Define:
   Fair Gamble- If expected value = cost of gamble.

   a. Is lottery Fair Gamble?
   b. Is Las Vegas a Fair Gamble?
   c. Is a Fair Game a Fair Gamble?

C. *Attitude Toward Risk* can be defined in terms of a *Fair Gamble*.

1. If *risk averse*, reject a Fair Gamble.
   i.e., pick B over A.
   $1 with certainty is preferred to gamble;
   B.($1; 1) $\succ$ A.($2, $0; ½).

   a. Risk aversion implies $U''(W) < 0$;
   - diminishing MU(W);
   - first $ is valued more than second $.

   Proof: Max. E(U) implies:

   E(U) of B > E(U) of A;

   U($1) > ½ U($2) + ½ U($0);

   Multiply both sides by 2:

   2 U($1) > U($2) + U($0);

   U($1) – U($0) > U($2) – U($1);

   b. For risk averse investor,
   disutility of $1 loss > utility of $1 gain.

2. If *risk neutral*, indifferent between A & B;
   - between $1 with certainty or Fair Gamble;
   - B($1; 1) = A($2, $0; ½).

   a. Risk neutrality implies U"(W) = 0;
      - constant MU(W);
      - first $ is valued same as second $.
      - same proof, replace > with =.

   b. For risk neutral investor,
      disutility of $1 loss is = utility of $1 gain.

3. If *risk loving*, prefer a Fair Gamble;
   - pick A over B;
   - Fair Gamble is preferred to $1 with certainty;
   - B.($1; 1) $\prec$ A.($2, $0; ½).

   a. Risk loving implies U"(W) > 0;
      - increasing MU(W);
      - first $ is valued less than second $.
      - same proof, replace > with <.

   b. For risk loving investor,
      disutility of $1 loss is < utility of $1 gain.

D. *Attitude Toward Risk* can be shown
in terms of *Indifference Curves*.

1. Graphs of utility functions
display preference structure in [W, U(W)] space.

   However, portfolio choice problem is to
Max E(U) subject to available opportunities.

   To do this, need to consider indifference curves
in relation to opportunities locus (constraints);
- find point of tangency.

2. Indifference Curves plotted in [E(R), σ(R)] space;

   Combin. of [E(R), σ(R)] so that E(U) is constant;

   Show tradeoff between E(R) and σ(R)
for which investor is indifferent.

3. Example: Consider quadratic utility function again.

$$U(W) = aW - bW^2$$

a. Re-define U in terms of return, R.
[analysis holds for both U(W) and U(R)].

$$U(R) = aR - bR^2$$

Thus $E[U(R)] = a\,E(R) - b\,E(R^2)$

b. Totally differentiate:

$dU = d[U(R)] = a\,dR - 2bR\,dR = (a - 2bR)\,dR$

[Intuition: Recall, $MU(R) = dU/dR = a - 2bR$.]
[Multiply both sides by dR.]

c. Then Expected Utility is $f[E(R), \sigma(R)]$.

Proof: First, recall definition of σ^2 -

$$\begin{aligned}\sigma^2(R) &= E[R - E(R)]^2 \\ &= E[R - E(R)][R - E(R)] \\ &= E[R^2 - 2\,R{*}E(R) + E(R)^2] \\ &= E(R^2) - 2\,E(R){*}E(R) + E(R)^2 \\ &= E(R^2) - E(R)^2\end{aligned}$$

Then, consider E(U):

$$\begin{aligned}E(U) &= a\,E(R) - b\,E(R^2) \\ &= a\,E(R) - b\,E(R^2) - b\,E(R)^2 + b\,E(R)^2 \\ &= a\,E(R) - b\,E(R)^2 - b\,[E(R^2) - E(R)^2] \\ &= a\,E(R) - b\,E(R)^2 - b\,\sigma^2(R)\end{aligned}$$

Thus: $E(U) = f[E(R), \sigma(R)]$;

If E(R) increases, E(U) increases;

For risk averse investor, $b > 0$, so that,
If $\sigma(R)$ increases, E(U) decreases.

d. Given the quadratic utility function,
can construct indifference curves in [E(R), σ(R)] space.

* Totally differentiate E(U), and hold E(U) constant;
gives tradeoff between E(R) & σ(R) holding E(U) constant.

We have: $E(U) = a\,E(R) - b\,E(R)^2 - b\,\sigma^2(R)$

Then: $d[E(U)] = a\,[dE(R)] - 2b\,E(R)\,[dE(R)] - 2b\,\sigma(R)\,[d\sigma(R)]$

Set d[E(U)] = 0 (indifferent!); solve for slope of indiff. curve:

$$0 = (a - 2b\,E(R))\,[dE(R)] - 2b\,\sigma(R)\,[d\sigma(R)]$$

$$(a - 2b\,E(R))\,[dE(R)] = 2b\,\sigma(R)\,[d\sigma(R)]$$

e. slope of indiff. curve $= \dfrac{[dE(R)]}{[d\sigma(R)]} = \dfrac{2b\,\sigma(R)}{a - 2b\,E(R)}$

Recall:

i. $U'(R) = MU(R) = a - 2bR > 0$ (Denominator > 0).

ii. Numerator depends on b (σ(R) > 0).
1) risk averse, $b > 0$, slope > 0;
2) risk neutral, $b = 0$, slope = 0;
3) risk loving, $b < 0$, slope < 0.

f. concavity of indiff. curve $= \dfrac{[d^2E(R)]}{[d\sigma(R)^2]} = \dfrac{2b}{a - 2b\,E(R)}$

i. risk averse, $b > 0$, concave upward;
ii. risk neutral, $b = 0$, straight line;
iii. risk loving, $b < 0$, concave downward.

g. σ(R) is also in numerator of slope;
Thus, if σ(R) = 0, slope = 0;
i.e., when curve hits vertical axis, it is horizontal.

E. Other aspects of *Attitude Toward Risk*.

First property: U'(W) = MU(W) > 0.

Second property: U"(W); attitude toward risk.

Third, consider how the total $ amount invested in risky assets changes as Wealth increases.

Investor may diversify; put some of wealth (X) in risky assets, and the rest (W-X) in the riskfree asset.

If Wealth increases, will investor put more or less of the additional wealth into risky assets?
i.e., will X increase, stay the same, or decrease?

1. *Absolute Risk Aversion* (ARA).
   Given an increase in Wealth:
   i. if investor increases X, decreasing ARA.
   ii. if investor keeps X same, constant ARA.
   iii. if investor decreases X, increasing ARA.

   If i, as W increases, investor increases X, "less averse to risk" in absolute terms.

   Most investors display decreasing ARA; as W increases, invest more $ (X) in risky assets.

   If investors can identify their feelings toward ARA, then the # of investment options can be reduced.

   Furthermore, which assumption is imposed will restrict the possible forms of utility functions that could describe their preferences.

   The Pratt-Arrow measure of ARA is:

   $$ARA = -U''(W) / U'(W).$$

   - measures nature of ARA for a given level of W.
   - examples later.

Fourth, consider how the amount invested in risky assets changes as a % of Wealth, as Wealth increases.

If Wealth increases, will investor put a greater or smaller % of that additional wealth into risky assets?
i.e., will (X/W) increase, stay the same, or decrease?

2. *Relative Risk Aversion* (RRA).
   Given an increase in Wealth:
   i. if investor increases (X/W), decreasing RRA.
   ii. if investor keeps (X/W) same, constant RRA.
   iii. if investor decreases (X/W), increasing RRA.

   If i, then as W increases, investor increases (X/W), "less averse to risk" relative to total wealth.

   Typical investor probably displays constant RRA; as W increases, invest same % (X/W) in risky assets.

   If we multiply the Pratt-Arrow measure of ARA by W, we get the measure of RRA:

$$RRA = -W * [U''(W) / U'(W)]$$

POINT: Utility Theory is important.

Even if the portfolio manager doesn't believe in
deriving utility functions,
can still learn from utility analysis.

This provides background for the theory of
making financial decisions under uncertainty;
investment theory is based on this!

Also, understanding properties of utility functions
can give insight into the process of rational choice.

This allows investor to eliminate some portfolios
from further consideration, and reduces
the chance of making a really bad decision.