

**Elton, Gruber, Brown, and Goetzmann**  
***Modern Portfolio Theory and Investment Analysis*, 7th Edition**  
**Solutions to Text Problems: Chapter 21**

Chapter 21: Problem 1

We can use the cash flows bonds A and B to replicate the cash flows of bond C. Let  $Y_A$  be the fraction of bond A purchased and  $Y_B$  be the fraction of bond B purchased. (Note that these are *not* investment weights that sum to 1.) Then we have:

$$t = 1: \quad \$100 Y_A + \$80 Y_B = \$90$$

$$t = 2: \quad \$1,100 Y_A + \$1,080 Y_B = \$1,090$$

Solving the above two equations simultaneously gives  $Y_A = Y_B = 1/2$ . So buying 1/2 of bond A and 1/2 of bond B gives the same cash flows as buying 1 bond C (or, equivalently, buying 1 bond A and 1 bond B gives the same cash flows as buying 2 of bond C). Therefore, if the Law of One Price held, the bonds' current prices would be related as follows:

$$1/2 P_A + 1/2 P_B = P_C$$

But, since we are given that  $P_A = \$970$ ,  $P_B = \$936$  and  $P_C = \$980$ , we have instead:

$$1/2 \times \$970 + 1/2 \times \$936 = \$953 < \$980$$

The Law of One Price does not hold.

Given that the future cash flows of the portfolio of bonds A and B are identical in timing and amount to those of bond C, and assuming that all three bonds are in the same risk class, an investor should purchase 1 bond A and 1 bond B rather than 2 of bond C.

Chapter 21: Problem 2

A.

A bond's current yield is simply its annual interest payment divided by its current price, so we have:

$$\text{Current Yield} = \$100 \div \$960 = 0.1042 \text{ (10.42\%)}$$

B.

A bond's yield to maturity is the discount rate that makes the sum of the present values of the bond's future cash flows equal to the bond's current price. Since this bond has annual cash flows, we need to find the rate,  $y$ , that solves the following equation:

$$\$960 = \sum_{t=1}^5 \left( \frac{\$100}{(1+y)^t} \right) + \frac{\$1000}{(1+y)^5}$$

We can find  $y$  iteratively by trial and error, but the easiest way is to use a financial calculator and input the following:

$$PV = -960$$

$$PMT = 100$$

$$FV = 1000$$

$$N = 5$$

After entering the above data, compute  $I$  to get  $I = y = 11.08\%$ .

### Chapter 21: Problem 3

In general, the nominally annualized spot rate for period  $t$  ( $S_{0t}$ ) is the yield to maturity for a  $t$ -period zero-coupon (pure discount) instrument:

$$P_0 = \frac{F}{\left(1 + \frac{S_{0t}}{2}\right)^t}$$

where  $P_0$  is the zero's current market price,  $F$  is the zero's face (par) value, and  $t$  is the number of *semi-annual* periods left until the zero matures.

The zero-coupon bonds in this problem all have face values equal to \$1,000.

If semi-annual periods are assumed, then bond A is a one-period zero, bond B is a two-period (one-year) zero, bond C is a three-period zero, and bond D is a four-period (two-year) zero.

So we have:

$$960 = \frac{1000}{\left(1 + \frac{S_{01}}{2}\right)^1} \Rightarrow S_{01} = 0.0833 \text{ (8.33\%)}$$

$$920 = \frac{1000}{\left(1 + \frac{S_{02}}{2}\right)^2} \Rightarrow S_{02} = 0.0851 \text{ (8.51\%)}$$

$$885 = \frac{1000}{\left(1 + \frac{S_{03}}{2}\right)^3} \Rightarrow S_{03} = 0.0831 \text{ (8.31\%)}$$

$$855 = \frac{1000}{\left(1 + \frac{S_{04}}{2}\right)^4} \Rightarrow S_{04} = 0.0799 \text{ (7.99\%)}$$

The nominally annualized implied forward rates ( $f_{t,t+j}$ ) can be obtained from the above spot rates. A general expression for the relationship between current spot rates and implied forward rates is:

$$f_{t,t+j} = \left[ \frac{\left( \left( 1 + \frac{S_{0,t+j}}{2} \right)^{t+j} \right)^{\frac{1}{j}}}{\left( 1 + \frac{S_{0,t}}{2} \right)^t} - 1 \right] \times 2$$

where  $t$  is the semi-annual period at the end of which the forward rate begins,  $j$  is the number of semi-annual periods spanned by the forward rate, and both  $t$  and  $j$  are integers greater than 0.

We can obtain a set of one-period forward rates by setting  $j$  equal to 1 and varying  $t$  from 1 to 3 in the preceding equation:

$$f_{12} = \left[ \frac{\left(1 + \frac{S_{02}}{2}\right)^2}{\left(1 + \frac{S_{01}}{2}\right)^1} - 1 \right] \times 2 = \left( \frac{(1.0426)^2}{(1.0417)^1} - 1 \right) \times 2 = 0.0870 \text{ (8.70\%)}$$

$$f_{23} = \left[ \frac{\left(1 + \frac{S_{03}}{2}\right)^3}{\left(1 + \frac{S_{02}}{2}\right)^2} - 1 \right] \times 2 = \left( \frac{(1.0416)^3}{(1.0426)^2} - 1 \right) \times 2 = 0.0792 \text{ (7.92\%)}$$

$$f_{34} = \left[ \frac{\left(1 + \frac{S_{04}}{2}\right)^4}{\left(1 + \frac{S_{03}}{2}\right)^3} - 1 \right] \times 2 = \left( \frac{(1.0400)^4}{(1.0416)^3} - 1 \right) \times 2 = 0.0704 \text{ (7.04\%)}$$

If instead we wanted the expected spot yield curve one period from now under the pure expectations theory, we can set  $t$  equal to 1 and vary  $j$  from 1 to 3 in the preceding equation:

$$\bar{S}_{12} = f_{12} = \left[ \frac{\left(1 + \frac{S_{02}}{2}\right)^2}{\left(1 + \frac{S_{01}}{2}\right)^1} - 1 \right] \times 2 = \left( \frac{(1.0426)^2}{(1.0417)^1} - 1 \right) \times 2 = 0.0870 \text{ (8.70\%)}$$

$$\bar{S}_{13} = f_{13} = \left[ \frac{\left(1 + \frac{S_{03}}{2}\right)^{\frac{3}{2}}}{\left(1 + \frac{S_{01}}{2}\right)^1} - 1 \right] \times 2 = \left[ \frac{(1.0416)^{\frac{3}{2}}}{(1.0417)^1} - 1 \right] \times 2 = 0.0831 \text{ (8.31\%)}$$

$$\bar{S}_{14} = f_{14} = \left[ \frac{\left(1 + \frac{S_{04}}{2}\right)^{\frac{4}{3}}}{\left(1 + \frac{S_{01}}{2}\right)^1} - 1 \right] \times 2 = \left[ \frac{(1.0400)^{\frac{4}{3}}}{(1.0417)^1} - 1 \right] \times 2 = 0.0789 \text{ (7.89\%)}$$

## Chapter 21: Problem 4

We can use the cash flows bonds A and B to replicate the cash flows of bond C. Let  $Y_A$  be the fraction of bond A purchased and  $Y_B$  be the fraction of bond B purchased. Then we have:

$$t = 1: \quad \$80 Y_A + \$1,100 Y_B = \$120$$

$$t = 2: \quad \$1,080 Y_A + \$0 Y_B = \$1,120$$

Solving the above two equations simultaneously gives:

$$Y_A = \frac{1,120}{1,080} = \frac{28}{27} = \frac{308}{297}$$
$$Y_B = \frac{\left(120 - 80 \times \frac{28}{27}\right)}{1,100} = \frac{\left(\frac{3,240}{27} - \frac{2,240}{27}\right)}{\frac{29,700}{27}} = \frac{1,000}{27} \times \frac{27}{29,700} = \frac{1,000}{29,700} = \frac{10}{297}$$

So buying  $308/297$  of bond A and  $10/297$  of bond B gives the same cash flows as buying 1 bond C (or, equivalently, buying 308 of bond A and 10 of bond B gives the same cash flows as buying 297 of bond C). Therefore, if the Law of One Price held, the bonds' current prices would be related as follows:

$$308/297 P_A + 10/297 P_B = P_C$$

But, since we are given that  $P_A = \$982$ ,  $P_B = \$880$  and  $P_C = \$1,010$ , we have instead:

$$308/297 \times \$982 + 10/297 \times \$880 = \$1,048 > \$1,010$$

The Law of One Price does not hold. For the Law of One Price to hold, bond C would have to sell for \$1,048.

## Chapter 21: Problem 5

If the Law of One Price holds, then the same discount rate (which is a spot rate) applies for the cash flows in a particular period for all three bonds. Also, in the presence of taxes, the price of each bond must equal the sum of the present values of its future after-tax cash flows, where the present values are calculated using the spot rates.

Each bond's capital gain or loss is simply its principal (par) value minus its price. Given that each bond has a par value of \$1,000, bond A has a capital gain of  $\$1,000 - \$985 = \$15$ , bond B has a capital gain of  $\$1,000 - \$900 = \$100$ , and bond C has a capital loss of  $\$1,000 - \$1,040 = -\$40$ .

Given that the periods shown are annual, that taxes must be paid on capital gains and can be deducted on capital losses, and that the capital gain or loss tax rate is one-half of the ordinary income tax rate, we need to find the discount factors and ordinary income tax rate that makes the following set of equations hold simultaneously:

$$\$80 \times (1 - T) \times d_2 + \$80 \times (1 - T) \times d_4 - \$15 \times \frac{T}{2} \times d_4 + \$1,000 \times d_4 = \$985$$

$$\$100 \times (1 - T) \times d_2 - 100 \times \frac{T}{2} \times d_2 + \$1,000 \times d_2 = \$900$$

$$\$120 \times (1 - T) \times d_2 + \$120 \times (1 - T) \times d_4 + \$40 \times \frac{T}{2} \times d_4 + \$1,000 \times d_4 = \$1,040$$

where

$T$  = the ordinary income tax rate;

$d_2 = \frac{1}{\left(1 + \frac{S_{02}}{2}\right)^2}$  = the two-semi-annual-period (one-year) discount factor;

$d_4 = \frac{1}{\left(1 + \frac{S_{04}}{2}\right)^4}$  = the four-semi-annual-period (two-year) discount factor.

The solution to the above set of simultaneous equations is:

$$T = 0.3303; d_2 = 0.8568; d_4 = 0.8934$$